Energy Conservation and Poynting Theorem in Electromagnetics: A Conceptual Perspective

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Energy conservation is an important principle in the physical sciences. In this tutorial, this principle, with specific reference to electromagnetism, is presented from a conceptual perspective, essentially drawing from ideas in [1-6]. In the context of electromagnetic power flow, it is Poynting theorem that represents conservation principle, and as such this theorem is of fundamental significance. The Poynting equation was derived by Poynting in the year 1884; later in the same year Heaviside too derived the same equation [1]. Considering the fact that Maxwell made rich contributions to electromagnetic theory, it may be noted here that though he derived several equations for the static energies in electric and magnetic fields, he did not consider power flow in electromagnetic fields [7].

Broadly speaking, the principle of energy conservation says that the total energy in the world, assuming it is closed, is unchanged. In this sense, it is a universal law. But how do we explain that the total energy is conserved? One way is by *localizing* this law by stating that if the amount of energy in a given volume increases or decreases, it is because energy is flowing into or out of the boundaries of that volume. Thus the law of energy conservation apparently involves two ideas [2]:

(A) The total energy in the world is a constant;

(B) Any change in the amount of energy in a volume of space has to be associated with flow of energy through the boundaries of the volume.

Since the ideas of conservation in respect of circuits and charges will be helpful in discussing electromagnetic energy conservation, we undertake a discussion on them first. A quick recollection of some fundamental terms will be helpful in reviewing the concepts, and hence we briefly state them before proceeding:

Force: 'Any action that tends to maintain or alter the motion of a body or to distort it' [8]. SI unit is Newton (N).

Work: An activity in which a force acts on an object and displaces in its direction. SI unit is Newton metre (N.m) or Joules (J).

Energy: The ability for doing work. SI unit is Joules, as work and energy are equivalent concepts. The related concept of energy density represents energy stored per unit volume, with the unit of J/m^3.

Power: The rate of doing work or the rate of using energy. SI unit is Joules per second, or watts. In electromagnetics, power density is defined as power per unit area, having the unit of W/m^2.

Energy conservation in circuits:

In electrical circuit theory, the power *p* supplied through a pair of terminals is given by *p* = *vi*, where *v* is the voltage between the terminals and *i* is the current flowing into/out of the appropriate terminal. Let's consider the electric circuit shown in Fig. 1. For this circuit, using Kirchhoff's laws, it can be shown that [3]

$$vi = \frac{d}{dt}\left[\frac{1}{2}Cv^2 + \frac{1}{2}Li_L^2\right] + \frac{v^2}{R} \tag{1}$$

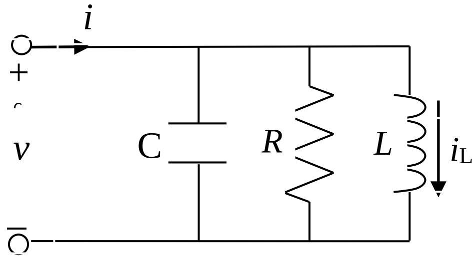

Fig. 1. An electric circuit in which the input power is conserved as stored energies in *C* and *L* and as heat energy in *R*.

In (1), the first and second terms within square brackets on the right side are recognized as terms representing energy stored in the capacitor and in the inductor respectively. The time-derivative makes the whole second term represent power. The second term is power dissipated in the resistor in the form of heat.

Thus (1) says that the total electrical power supplied to the circuit is equal to the electrical and magnetic energies stored in the reactive elements plus the energy dissipated in the resistor in the form of heat. It is important to note that it is energy in general, and not electrical energy, that is conserved [3]. Therefore, we need to appropriately modify conservation idea (B) above as follows: Any change in the amount of energy in a volume of space has to be associated with either flow of energy through the boundaries of the volume and/or dissipation or generation of energy within the volume..

Electric charge conservation:

A fundamental implication of the general law of conservation of energy in electrical science is that the total charge is conserved. This means that charge can neither be created not destroyed but can only be transferred from one place to another.

While charge conservation in a broad sense is a universal principle, its transfer is necessarily a 'local' process. Thus, if the charge density ρ in a certain lossless volume is decreasing, this can only mean that charge is flowing away through the surface bounding the volume. If we represent the rate of this 'flow' by current density **J**, then this statement can be mathematically represented as

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (2)$$

Now, charge is matter. Both the quantities in (2), ρ and **J**, are defined by this matter. As we know, accelerated charges radiate electromagnetic fields. Can we then extend the idea of charge conservation to the idea of conservation of energy of the electromagnetic field?

Conservation in electromagnetic fields:

Towards considering this question, since electromagnetic fields spread through space, we conceptualize electromagnetic power as flowing through all of space. This means we should be able to assign each point in space an appropriate power or energy term. Therefore, we define *point* forms of relevant field quantities - in this case energy density W (J/m^3), power dissipation density P_d (W/m^3) and power density vector **S** (W/m^2). Consistent with our notion of localizing energy conservation, we should in practice consider power flowing into or out of a volume of space through the surface enclosing the volume. A consideration of total power flow would then include integration of appropriate quantities over surface and volume.

With reference to Fig. 2, let W be the energy density due to electric and magnetic fields inside volume V enclosed by surface S. Defining da as differential surface area on S with unit normal $\hat{\mathbf{n}}$, let $\mathbf{S}$ be the power density (flow of this energy per unit time per unit area) normal to the surface. Then, analogous to the charge conservation equation (2), we would expect to be able to write for the electromagnetic field [2]

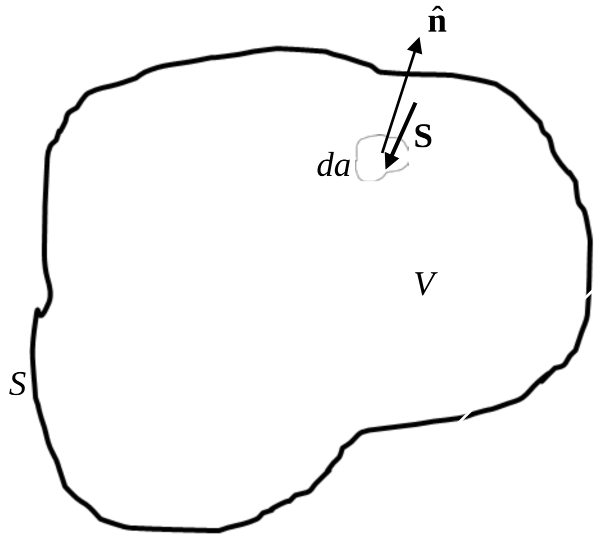

Fig. 2. Electromagnetic power flow into volume V through surface S.

$$\nabla \cdot \mathbf{S} = -\frac{\partial W}{\partial t} \qquad (3)$$

Equation (3) would be correct if the electromagnetic field energy by itself was conserved. But what if field energy, like electrical energy, is not conserved by itself? To consider this possibility, let us consider an antenna connected to a transmitter circuit. While the antenna does not radiate electromagnetic energy when the transmitter is off, when it is turned on, accelerated charges flow into the antenna and it emits EM field energy (as per Maxwell's equations). Thus, because an accelerated charge (matter) emits electromagnetic field (which is not matter), and thus lose its energy to the field, in the case of electromagnetic energy conservation, we need to account for both the current (source) and the field. Thus electromagnetic field energy *alone*, just like electrical energy, is not conserved. Therefore, (3) needs to be modified to be applicable in this case.

A complete conservation law then requires a consideration of the effect of matter, in this case the charges. The field energy is affected either when the field expends energy on the matter or the matter expends energy on the field. If we describe this by power dissipation P_d, then (3) would modify as [2, 3]

$$\nabla \cdot \mathbf{S} = -\frac{\partial W}{\partial t} - P_d \qquad (4)$$

When **S**, W and P_d are appropriately defined, an equation in the form of (4) can describe the interrelationship between flow, storage and dissipation of electromagnetic energy [2]. We note that while **S** and W correspond to the fields, P_d represents the interaction between charges and fields. It will help to obtain an explicit expression for this term in terms of the electric field **E** and the source **J** before proceeding. To this end, we start with the force on a particle of charge q moving with a velocity **v** in the presence of electric field **E** and magnetic field **B**. This force is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (5)$$

Since work done by the force on the charge is given by the dot product of force and displacement, the rate of doing work P_d is given by

$$P_d = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} + q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v} \qquad (6)$$

Since $\mathbf{v} \times \mathbf{B}$ is necessarily perpendicular to **v**, the term $q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}$ in (6) vanishes. If there are N particles per unit volume, the energy loss per unit volume per unit time then is

$$P_d = Nq\mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \mathbf{J} \qquad (7)$$

where we have employed the fact that $\mathbf{J} = Nq\mathbf{v}$. Now, (7) can represent either (a) power dissipated in a conductor if **E** is the field strength required to produce **J**, or (b) kinetic and potential energy imparted by **E** on the charged carriers present within the volume, or (c) power used in establishing a current **J** against the field **E**. If **E** and **J** are oppositely directed, as in (c), then (7) gives negative power meaning the source is generating electric power. By using (7), we can now rewrite (4) as

$$\nabla \cdot \mathbf{S} = -\frac{\partial W}{\partial t} - \mathbf{E} \cdot \mathbf{J} \qquad (8)$$

It should be noted that magnetic field does no work on charge, and therefore an equation similar to (7) for magnetic field is not warranted.

Poynting theorem for electromagnetic power flow:

Starting from Maxwell's curl equations, and employing certain mathematical manipulations, it is straightforward to obtain the following point-function relationship [5]:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) - \mathbf{E} \cdot \mathbf{J} \qquad (9)$$

We may observe that (9) is of the same form as (8), with the term **E**·**J**, representing power dissipated by the field on the charges, appearing in both. We also recollect that, in (9), the term $\frac{1}{2}\varepsilon E^2$ is considered as representing the energy density of the electric field and the term $\frac{1}{2}\mu EH^2$ that of the magnetic field. While these terms were originally derived for static fields, they can also be assumed to hold for dynamic fields [4]. Thus, the first term on the right in (9), just like the corresponding term in (8), represents the total stored energy per unit volume. With the two terms on the right in (9) thus proving to be consistent with the conservation equation (8), it follows by comparing the left hand sides of (8) and (9) that the term **S** = **E** × **H**, having units of watts per square meter, should represent the rate of energy flow per unit area at a given point.

An integral form of (9) can be obtained by integrating the left and right sides of the equation over the volume:

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = \int_V \nabla \cdot \mathbf{S} dv = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) dv - \int_V \mathbf{E} \cdot \mathbf{J} dv \qquad (10)$$

At the time of discussing the general conservation idea, we stated that power flow through a surface should balance the stored energy and the power dissipated within the volume enclosed by the surface. In order to mathematically express this statement, we can convert the volume integral on the left side of (10) into a surface integral by applying divergence theorem, and obtain the following equation:

$$\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_V \mathbf{E}\cdot\mathbf{J}dv \qquad (11)$$

The quantity **S** = **E** × **H**, the power density associated with the electromagnetic field, is called Poynting vector. The direction of **S**, which is normal to the plane containing **E** and **H**, defines the direction of the power flow. The statement given by the above equation that the *surface integral* of this quantity over a closed surface equals the net power leaving the volume is referred to as Poynting theorem, so named after Poynting, the proposer of the idea. (11) may be rewritten as

$$-\oint_S \mathbf{S}\cdot d\mathbf{s} = \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dv + \int_V \mathbf{E}\cdot\mathbf{J}dv \qquad (12)$$

This equation (12) states that 'the total power flowing into a closed surface at any instant equals the sum of the rates of increase of the stored electric and magnetic energies and the ohmic power dissipated within the enclosed volume' [2]. Yaghjian [9] has shown that the integral of the Poynting vector dotted into the outward normal to a closed surface that lies in free space is exactly equal to the instantaneous power leaving that closed surface.

The following points may be noted regarding the power flow:

(1) The power density vector as defined by **S** would be zero if either **E** or **H** is zero. On a perfect conductor since tangential electric field is zero, there cannot be power flow normal to, and hence into, the conductor [6].

(2) If conductivity $\sigma = 0$ in the volume, then **J** is zero and thus there is no power dissipation. In this case, the second term on the right of (12) vanishes [5].

(3) In static case, the rate of change in electric and magnetic energies is zero, and the Poynting theorem would read that total power flowing into a closed surface is equal to the power dissipated within the volume [5].

Examples of power flow:

To illustrate the nature of electromagnetic power flow, let us consider some examples [2]. In the first example, we consider a circular capacitor as shown in Fig. 3 being charged with an alternating current. Nearly uniform electric field is normally directed from the positive plate to the negative plate, and thus alternates in direction with the frequency of the exciting current. Since this changing electric field links any loop between the plates, a magnetic field is produced as per Maxwell's equations, as shown by dotted circle in the figure. Thus there is field energy density in the space between the plates caused by the varying electric and magnetic fields. What is the direction of the power flow that makes this change possible? We would think that the power flows along the wires that carries the current that charges the capacitor. But looking at the directions of **E** and **H**, we see that power flow, as per Poynting theorem, is directed towards the axis of the capacitor!

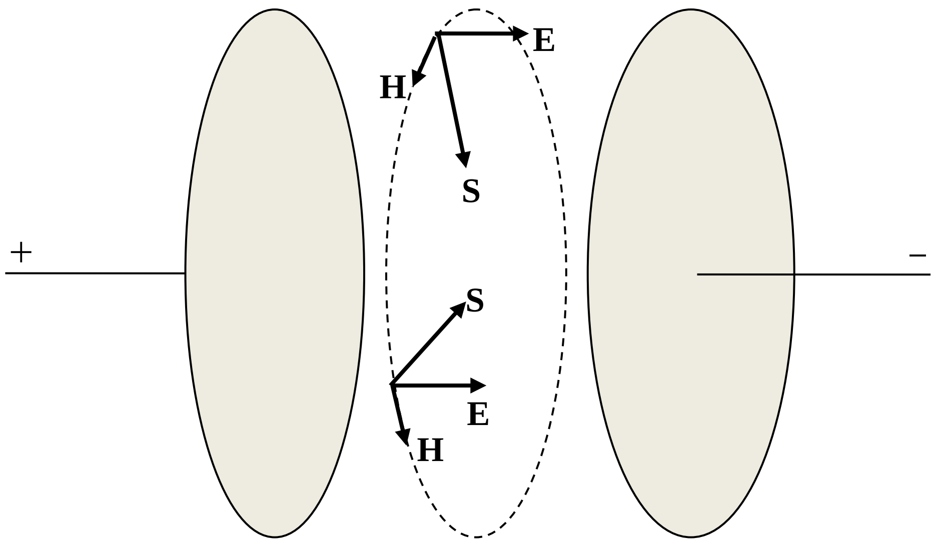

Fig. 3. Poynting vector in a charging capacitor. Redrawn based on [2].

Next let us consider a current carrying resistive wire, Fig. 4. Let us assume the current is established by an electric field along the resistive wire. Since the wire is resistive, a potential drop occurs along the conductor, in turn resulting in an external electric field parallel to the conductor surface.

Besides, the flowing current causes a magnetic field as well. Thus, the electric and magnetic field vectors are normal to each other. The implication, as seen from Fig. 4, is that power flow is towards the wire and all around it.

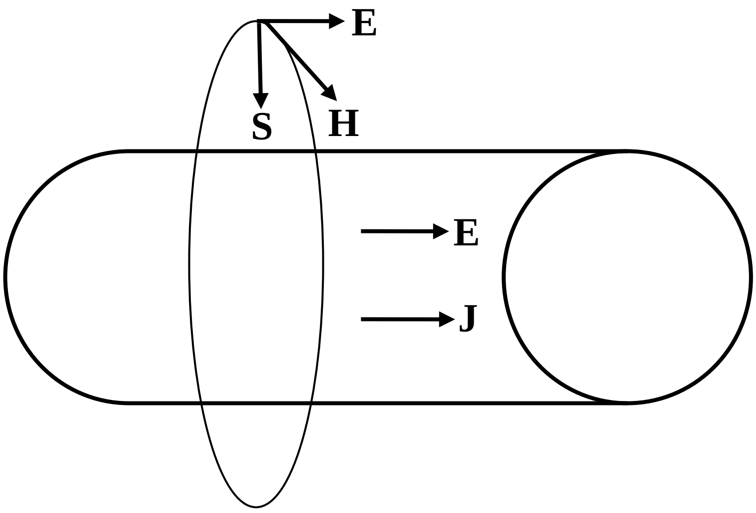

Fig. 4. Poynting vector in a resistive, current carrying wire. Redrawn based on [2].

As a final example, let us consider the hypothetical case of an electric charge placed near the centre of a bar magnet. Both the charge and the magnet being at rest, static electric and magnetic fields are directed as depicted in Fig. 5. Since they are normal to each other at every point, theory predicts a circulating power flow vector! However, when we integrate the flow around the closed surface that encloses the sources, we get a net flow of zero, as logically expected. Given that electrons are spinning inside a permanent magnet, as Feynman has noted [2], this outside circulation of energy is not conceptually strange.

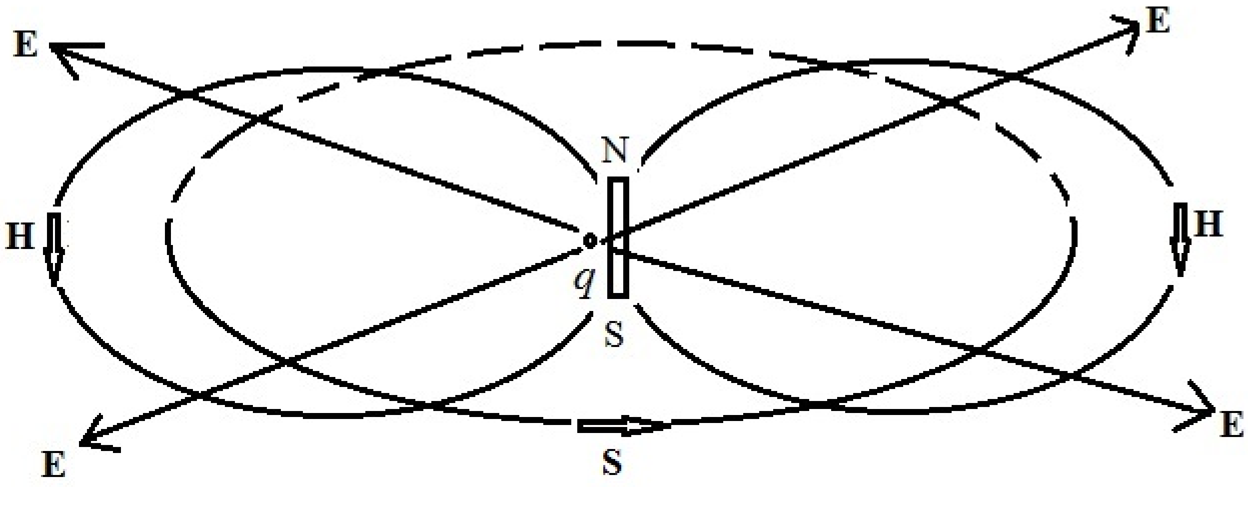

Fig. 5. Poynting vector of a magnet and an electric charge placed close to it. Redrawn based on [2].

Ambiguities with the power flow:

While the surface integral of **S** does represent the net power flow through a closed surface, as a point function, considering **S** as power density at *every* point is an arbitrary, but very useful, concept [1, 5]. That is, it is not possible to state where the field energy is located [3, 4]. As Stratton has noted [1], 'from a volume integral representing the total energy of a field no rigorous conclusion can be drawn with regard to its distribution.' This situation is very similar to gravitational energy, where, for example, one can easily calculate the potential energy of an object of mass *m* raised a height *h* above ground, but cannot say where exactly that energy resides. In Fig. 6, thus, the potential energy of the object is given by $W_p = mgh$. Clearly, one cannot assign a location for this energy.

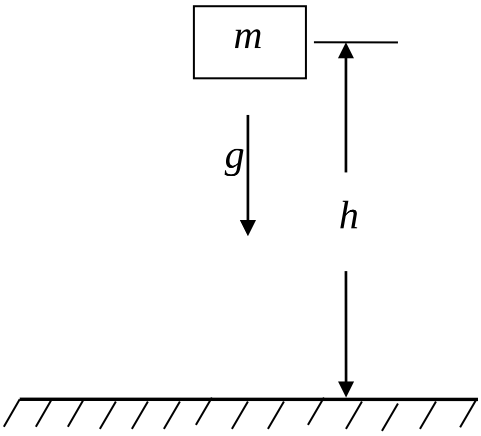

Fig. 6. An object of mass *m* at a height *h* from ground.

While we cannot thus say precisely where electromagnetic energy is located, it has been noted [2] that the solutions obtained for *W* and **S** themselves represent just one set of solutions, and several others are possible. ‘There are, in fact, an infinite number of different possibilities for W and **S**, and so far no one has thought of an experimental way to tell which one is right! People have guessed that the simplest one is probably the correct one, but we must say that we do not know for certain what is the actual location in space of the electromagnetic field energy.’ In fact, none of the various alternative forms of Poynting theorem that have been proposed has contributed anything of particular value to theory [1].

Conclusion:

Energy conservation principle is a very important fundamental idea in electromagnetics. This principle was reviewed in this paper, based on discussions in [1-6]. This topic, like several others including displacement current, offers an opportunity to discuss the conceptual richness of electromagnetics in a way that can profitably be employed in education [10, 11]. It was with this belief that this article has attempted to present the topic from a conceptual perspective.

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