Forum for Electromagnetic Research Methods and Application Technologies (FERMAT)

Frequency Mixing in Quasi-Periodic Stacks of Binary Nonlinear Layers

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***Abstract*— A generic semi-analytical approach is proposed for the self-consistent analysis of combinatorial frequency generation in stacks of binary nonlinear layers illuminated by a pair of pump waves. It is shown that the quasi-periodic stacks can be treated as the defected periodic structures with the defect located at the specific positions determined by a particular layer sequence. The developed technique combining the Harmonic Balance and Transfer Matrix Methods is illustrated by the cases of periodic and quasi-periodic (Fibonacci and Thue-Morse type) stacks of nonlinear dielectric layers.**

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A.

Modified Transfer Matrix Method for the Problems of Nonlinear Scattering by Periodic and Quasi-Periodic Layered Structures

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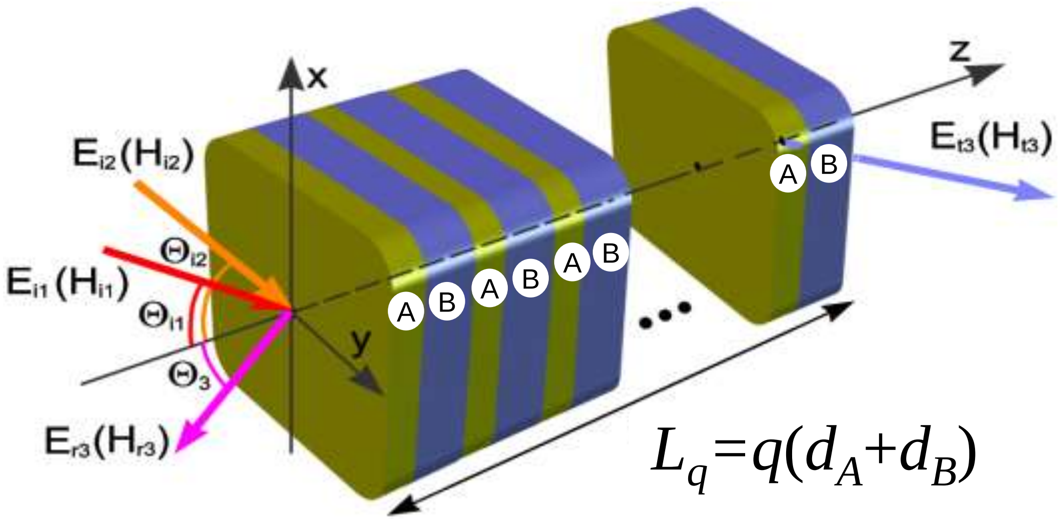
Outline

- Introduction
- Problem statement & assumptions
- Solution framework
- Periodic and quasi-periodic (Fibonacci and Thue-Morse) stacks
- Primitive cells and defects in Fibonacci and Thue-Morse stacks
- Simulation examples
- Concluding remarks

Periodic Stack

q unit cells with alternating *A* and *B*



$L_q = q(d_A + d_B)$

Electric displacement in nonlinear anisotropic dielectric layers:

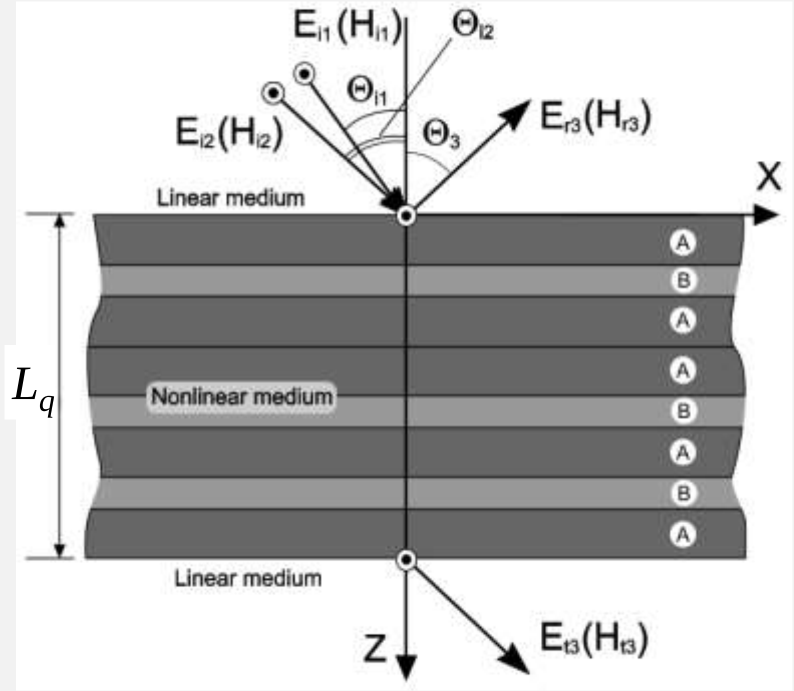
$$D_n^{A,B} = \varepsilon_0 \left(\varepsilon_{nm}^{A,B} + \chi_{nmk}^{A,B} E_k^{A,B} \right) E_m^{A,B}$$

$$\hat{\varepsilon}^{A,B} = \left(\varepsilon_{xx}, \varepsilon_{xx}, \varepsilon_{zz} \right)^{A,B} \qquad \hat{\chi}^{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & \chi_{xxz} & 0 \\ 0 & 0 & 0 & \chi_{xxz} & 0 & 0 \\ \chi_{zxx} & \chi_{zxx} & \chi_{zzz} & 0 & 0 & 0 \end{pmatrix}^{A,B}$$

Quasi-Periodic Stacks

Fibonacci stack of order q

$S_q=\{S_{q-1} \cup S_{q-2}\}, \; S_1=\{A\}, S_2=\{AB\}$

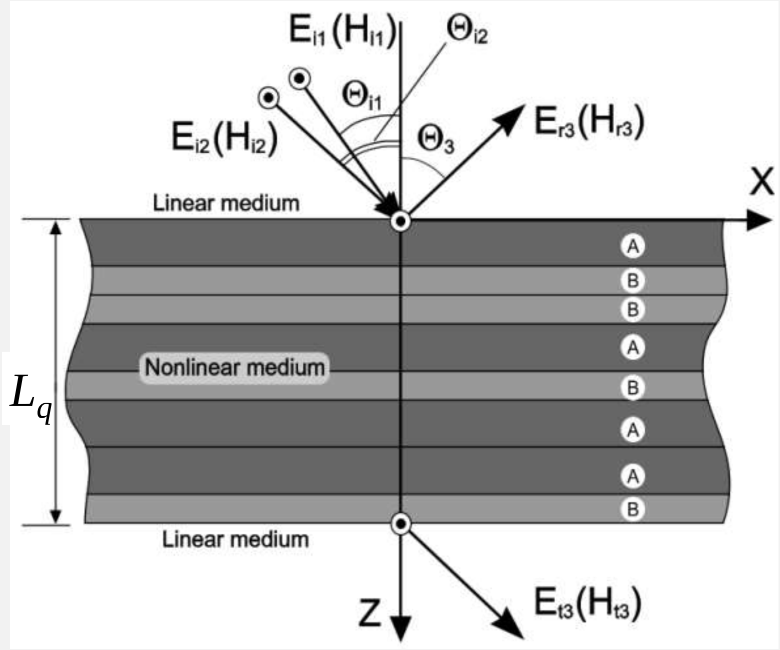


$$L_q=L_{q-1}+L_{q-2}$$
$$L_1=d_A, L_2= d = d_A + d_B$$

Thue-Morse stack of order q

$Q_q=\{Q_{q-1} \cup Q'_{q-1}\}$ & $Q'_q=\{Q'_{q-1} \cup Q_{q-1}\}$

$Q_0 = \{A\}, Q'_0 = \{B\}$



$$L_q=2L_{q-1}$$
$$L_1= d = d_A + d_B$$

Problem Statement for TM Waves

- Two TM plane waves of frequencies ω_1 and ω_2 are incident on the stacks at angles Θ_{i1} and Θ_{i2}
- Nonlinearity is assumed weak and the three-wave mixing process is dominant
- The problem is linearized by harmonic balance method to obtain combinatorial frequency fields
- In the non-depleting wave approximation the fields are determined recursively at each frequency
  - Pump waves: only reflected and refracted fields
  - Combinatorial frequencies: excited & scattered fields

Linearized Problem

In the three-wave mixing process, fields $H_y(\omega)$ of TM waves satisfy Helmholtz equation in each layer A and B at pump $\omega_{1,2}$ and combinatorial $\omega_3=\omega_1+\omega_2$ frequencies:

$$\left(\frac{\partial^2}{\varepsilon_{xx}\partial z^2}+k_p^2-\frac{k_{xp}^2}{\varepsilon_{zz}}\right)H_y\left(\omega_p\right)=\begin{cases}0, & p=1,2\\ 8\pi k_3\left[\frac{\partial}{\partial x}\left(\frac{\chi_{zxx}}{\varepsilon_{zz}}E_x\left(\omega_1\right)E_x\left(\omega_2\right)+\frac{\chi_{zzz}}{\varepsilon_{zz}}E_z\left(\omega_1\right)E_z\left(\omega_2\right)\right)-\right. & \\ \left.-\frac{\chi_{xxz}}{2\varepsilon_{xx}}\frac{\partial}{\partial z}\left(E_x\left(\omega_1\right)E_z\left(\omega_2\right)+E_x\left(\omega_2\right)E_z\left(\omega_1\right)\right)\right], & p=3\end{cases}$$

where

$$k_p=\omega_p/c,\ k_{x1,x2}=k_{1,2}\sqrt{\varepsilon_a}\sin\Theta_{i1,i2}$$

$$k_{x3}=k_{x1}+k_{x2}=k_3\sqrt{\varepsilon_a}\sin\Theta_3$$

Solution of the Linearized Problem

In a layer of type j=A,B in n^{th} primitive cell:

- At pump wave frequencies $\omega_{1,2}$

$$H_{yj}^{(n)}\left(\omega_p\right)=\left[\mathcal{B}_{pj}^{n+}e^{ik_{zj}^{(p)}z}+\mathcal{B}_{pj}^{n-}e^{-ik_{zj}^{(p)}z}\right]e^{-i\omega_p t+ik_{xp}x},\quad p=1,2$$

- At combinatorial frequency ω_3

$$H_{yj}^{(n)}\left(\omega_3\right)=\left(\mathcal{B}_{3j}^{n+}e^{ik_{zj}^{(3)}z}+\mathcal{B}_{3j}^{n-}e^{-ik_{zj}^{(3)}z}+D_{1j}^{n+}e^{ik_{zj}^{+}z}+D_{2j}^{n+}e^{-ik_{zj}^{+}z}+D_{1j}^{n-}e^{ik_{zj}^{-}z}+D_{2j}^{n-}e^{-ik_{zj}^{-}z}\right)e^{-i\omega_3 t+ik_{x3}x},$$

where $\mathcal{B}_{pj}^{n\pm}=\mathcal{B}_{j}^{n\pm}\left(\omega_p\right),\quad k_{zj}^{\pm}=k_{zj}^{(1)}\pm k_{zj}^{(2)},\quad k_{zj}^{(p)}=\sqrt{\varepsilon_{xxj}\left(k_p^2-k_{xp}^2/\varepsilon_{zzj}\right)},\quad p=1,2,3$

$$D_{1j}^{n+}=\alpha_j\beta_j\frac{\mathcal{B}_{1j}^{n+}\mathcal{B}_{2j}^{n+}}{\left(k_{zLj}^{+}\right)^2-\left(k_{zLj}^{(3)}\right)^2},\quad D_{2j}^{n+}=\alpha_j\beta_j\frac{\mathcal{B}_{1j}^{n-}\mathcal{B}_{2j}^{n-}}{\left(k_{zLj}^{+}\right)^2-\left(k_{zLj}^{(3)}\right)^2}$$

$$D_{1j}^{n-}=\alpha_j\gamma_j\frac{\mathcal{B}_{1j}^{n+}\mathcal{B}_{2j}^{n-}}{\left(k_{zLj}^{-}\right)^2-\left(k_{zLj}^{(3)}\right)^2},\quad D_{2j}^{n-}=\alpha_j\gamma_j\frac{\mathcal{B}_{1j}^{n-}\mathcal{B}_{2j}^{n+}}{\left(k_{zLj}^{-}\right)^2-\left(k_{zLj}^{(3)}\right)^2}$$

Amplitude Coefficients $\mathcal{B}_j^{n\pm}\left(\omega_p\right)$

The problem has been reduced to evaluating amplitudes of waves refracted into a layer of type $j = A, B$ in an n^{th} primitive cell of the stack

$$\mathcal{B}_j^{n\pm}\left(\omega_p\right) = \left(\eta_{j11}^{(n-1)}\left(\omega_p\right) \pm \frac{k_p}{k_{zLj}^{(p)}} \varepsilon_{xxj} \eta_{j21}^{(n-1)}\left(\omega_p\right)\right)\left[1 + R\left(\omega_p\right)\right] +$$

$$+ \frac{k_{za}^{(p)}}{k_p \varepsilon_a}\left(\eta_{j12}^{(n-1)}\left(\omega_p\right) \pm \frac{k_p}{k_{zLj}^{(p)}} \varepsilon_{xxj} \eta_{j22}^{(n-1)}\left(\omega_p\right)\right)\left[1 - R\left(\omega_p\right)\right]$$

where $R(\omega_p)$ - the reflection coefficient of the stack;

$\hat{\eta}_j^{(n-1)}\left(\omega_p\right)$ - the transfer matrices of a subset of (n - 1) primitive cells preceding the layer of type j in the n^{th} primitive cell.

Emission Coefficients

Amplitudes of the combinatorial frequency emission in the reverse (F_r) and forward (F_t) directions

$$F_r = \left(\frac{k_3 \varepsilon_a}{k_{za}^{(3)}} \widehat{\eta}_{N_q}(\omega_3)_{2,1} + \widehat{\eta}_{N_q}(\omega_3)_{2,2} \right) \lambda_1 - \left(\frac{k_3 \varepsilon_a}{k_{za}^{(3)}} \widehat{\eta}_{N_q}(\omega_3)_{1,1} + \widehat{\eta}_{N_q}(\omega_3)_{1,2} \right) \lambda_2,$$

$$F_t = -\left(\lambda_1 + \lambda_2 \frac{k_3 \varepsilon_a}{k_{za}^{(3)}} \right),$$

where $\widehat{\eta}_{N_q}(\omega_3)$ - a transfer matrix of the whole stack containing N_q primitive cells

$$\lambda_{1,2} = \sum_{n=1}^{N_q} f\left(\widehat{\eta}_j^{(n)}(\omega_3), D_{1j}^{n\pm}, D_{2j}^{n\pm} \right)$$

Transfer Matrices

- The transfer matrices of the stack and any its subset depend on
  - Stack configuration and layer sequence
  - Transfer matrices of the constituent layers
- The transfer matrices are to be calculated at each frequency: both pump waves and mixing products
- The transfer matrices at frequency ω_3 are used in the expression of $F_{r,t}$ and $\mathcal{B}_j^{n\pm}(\omega_3)$

Regular Periodic Stacks

- Transfer matrices for
  - Stack of *n* unit cell: $\widehat{\eta}_A^{(n)} = \widehat{\eta}_n = \left[\widehat{m}_{LA}(\omega_3)\widehat{m}_{LB}(\omega_3)\right]^n$
  - 1st layer in *n* unit cell stack: $\widehat{\eta}_B^{(n)} = \widehat{\eta}_{n-1}\widehat{m}_{LA}(\omega_3)$
  - Layers *A* and *B:* $\widehat{m}_{LA}(\omega_3)$ and $\widehat{m}_{LB}(\omega_3)$
- Transfer matrices $\widehat{\eta}_j^{(n)}$ have closed form relating $\widehat{\eta}_1$ with Bloch phase in periodic stacks
- There is no advantage of using the closed form here – fields must be calculated **inside** each layer
- The recursive relation for $\widehat{\eta}_n = \widehat{\eta}_{n-1}\left[\widehat{m}_{LA}(\omega_3)\widehat{m}_{LB}(\omega_3)\right]$

Quasi-Periodic Stacks

Transfer matrices for Fibonacci & Thue-Morse stacks are defined by the recursive relations

Fibonacci stacks of order $q \geq 2$

$S_q = \{S_{q-1} \cup S_{q-2}\}$, $S_1 = \{A\}$, $S_2 = \{AB\}$

Transfer matrix

$$\hat{M}_q(\omega_p) = \hat{M}_{q-1}(\omega_p)\hat{M}_{q-2}(\omega_p)$$

where

$$\hat{M}_0(\omega_p) = \hat{m}_{LB}(d_B, \omega_p),\ \hat{M}_1(\omega_p) = \hat{m}_{LA}(d_A, \omega_p);$$

Examples:

$d'_A = 2d_A$

$A'B$

$S_5 = \{AB\ AAB\ AB\ A\}$

$S_6 = \{AB\ AAB\ AB\ AAB\ AAB\}$

Thue-Morse stacks of order $q \geq 1$

$Q_q = \{Q_{q-1} \cup Q'_{q-1}\}$ & $Q'_q = \{Q'_{q-1} \cup Q_{q-1}\}$

$Q_0 = \{A\}$, $Q'_0 = \{B\}$

Transfer matrices

$$\hat{M}_q(\omega_p) = \hat{M}_{q-1}(\omega_p)\hat{M}'_{q-1}(\omega_p)$$

$$\hat{M}'_q(\omega_p) = \hat{M}'_{q-1}(\omega_p)\hat{M}_{q-1}(\omega_p)$$

$$\hat{M}_0(\omega_p) = \hat{m}_{LA}(d_A, \omega_p),\ \hat{M}'_0(\omega_p) = \hat{m}_{LB}(d_B, \omega_p)$$

Examples:

Defective cells

$Q_3 = \{AB\ BA\ BA\ AB\}$

$Q_4 = \{AB\ BA\ BA\ AB\ BA\ AB\ AB\ BA\}$

Quasi-Periodic Stacks

Fibonacci & Thue-Morse stacks can be treated like periodic stacks with defects

Fibonacci stacks

$S_5 = \{AB\ A'B\ AB\ A\}$

$S_6 = \{AB\ A'B\ AB\ A'B\ A'B\}$

- Two types of primitive cells:
  - regular $\{AB\}$
  - "defective" $\{A'B\}$
- Layer A' in "defective" cells is a doublet $A'=AA$ with $d'_A = 2d_A$
- Both "regular and "defective" cells have the layers in the same order

Thue-Morse stacks

$Q_3 = \{AB\ BA\ BA\ AB\}$

$Q_4 = \{AB\ BA\ BA\ AB\ BA\ AB\ AB\ BA\}$

- Two types of primitive cells:
  - regular $\{AB\}$
  - "defective" $\{BA\}$
- Layers A and B in "defective" cells are interchanged
- Both "regular and "defective" cells have the same thickness $d=d_A+d_B$

Positions of "defective" cells in the stacks are to be determined

Primitive Cells in Thue-Morse Stacks

- The number of primitive cells in a stack of order *q*:

$$N_q = 2^{q-1}$$

- The positions of the regular and defective cells are determined by their serial number *n* using the following recurrence relation at $n \geq 3$

$$t_n = \begin{cases} 1 - t_{n-1}, & n - \text{ even} \\ t_{\left(\frac{n-1}{2}+1\right)}, & n - \text{ odd} \end{cases}$$

where $t_n = 0$ for the regular cell $\{AB\}$
$t_n = 1$ for defective cells $\{BA\}$
$t_1 = 0$, $t_2 = 1$

Primitive Cells in Fibonacci Stacks

The number of primitive cells in a stack of order *q*:

$$N_q = \begin{cases} \dfrac{\Phi_{q+1} - \Gamma_q}{2}, & q - \text{ even} \\ \dfrac{\Phi_{q+1} - \Gamma_q - 1}{2}, & q - \text{ odd} \end{cases}$$

where $\Phi_q = \Phi_{q-1} + \Phi_{q-2}$ is Fibonacci number, $\Phi_1 = \Phi_2 = 1$

The number of "defective" cells Γ_q: $\Gamma_q = 0$ at $q \le 3$ and at $q \ge 4$

$$\Gamma_q = \begin{cases} \dfrac{\left(\dfrac{3+\sqrt{5}}{2}\right)^{\frac{q}{2}-1} - \left(\dfrac{3-\sqrt{5}}{2}\right)^{\frac{q}{2}-1}}{\sqrt{5}}, & q - \text{ even} \\ \Gamma_{q+1} - \Gamma_{q-1} - 1, & q - \text{ odd} \end{cases}$$

“Defective” Cells in Fibonacci Stacks

Positions of additional *A* layers in Fibonacci stack of order q are defined by a row-matrix $\widehat{P}_q$ of length Φ_{q+1} with 1’s in the columns for the first *A* layer of the doublets:

$$S_6 = \{AB\ AAB\ AB\ AAB\ AAB\}$$

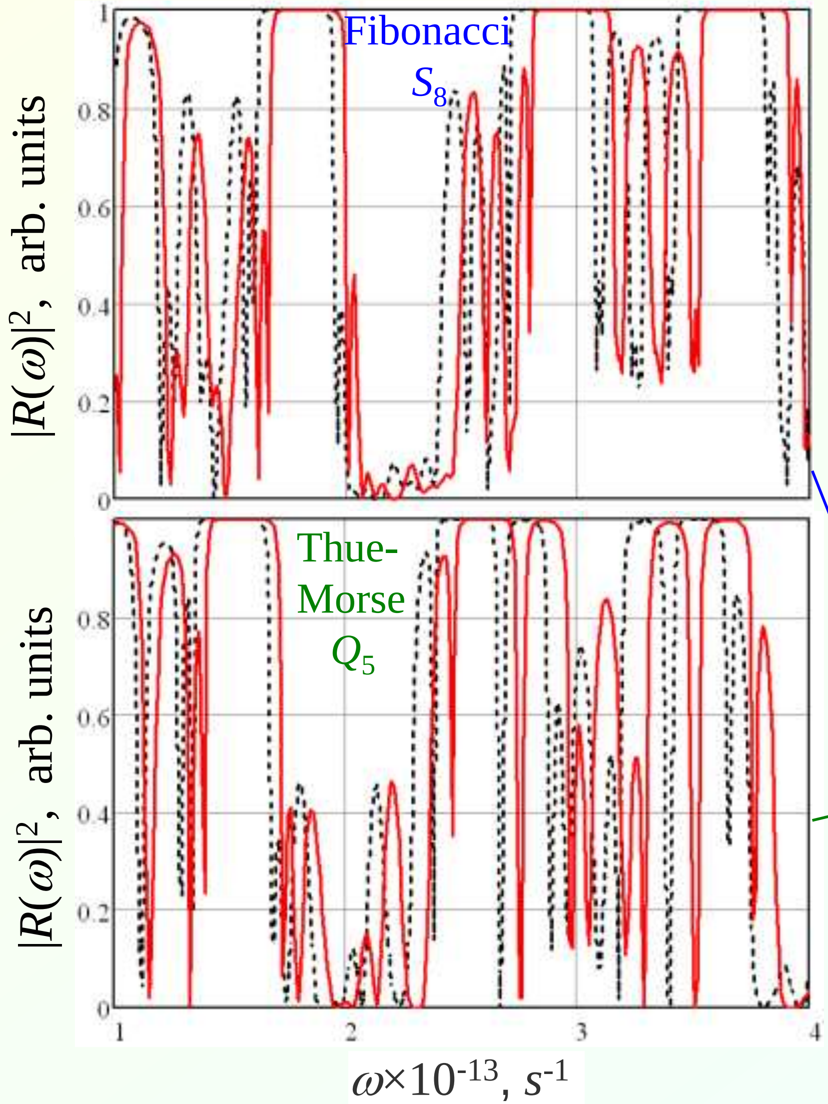
$$\widehat{P}_6 = [00\ 100\ 00\ 100\ 100]$$

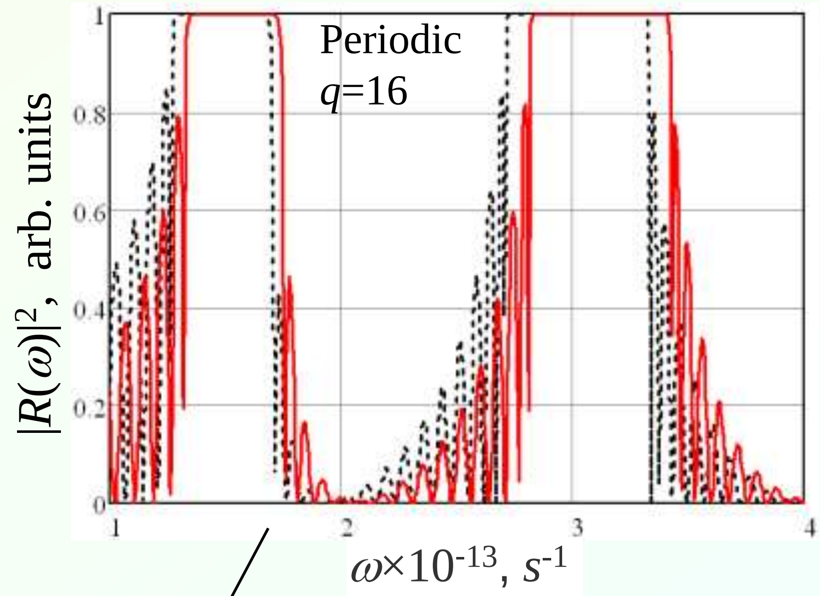
$\widehat{P}_q$ is defined by the recurrence relations:

$$\widehat{P}_q = \widehat{P}_{q-1} + \widehat{P}_{q-2}\widehat{\Psi}\left(\Phi_q\right) + \begin{cases} \widehat{u}\left(\Phi_q\right), & q-\text{even} \\ 0, & q-\text{odd} \end{cases}$$

where $\widehat{u}\left(\Phi_q\right) = \left\{\delta_{i,\Phi_q}\right\}$ is a row-matrix with 1’s in Φ_q column only; $\widehat{\Psi}\left(\Phi_q\right) = \left\{\delta_{i+\Phi_q,j}\right\}$ is a square Toeplitz matrix with 1’s only at the secondary diagonal offset for Φ_q from the main diagonal.

Stack Reflectance: QPS vs Periodic





Periodic q=16 (32 layers, d_B=13µm)

Fibonacci S_8 (34 layers, d_B=12µm)

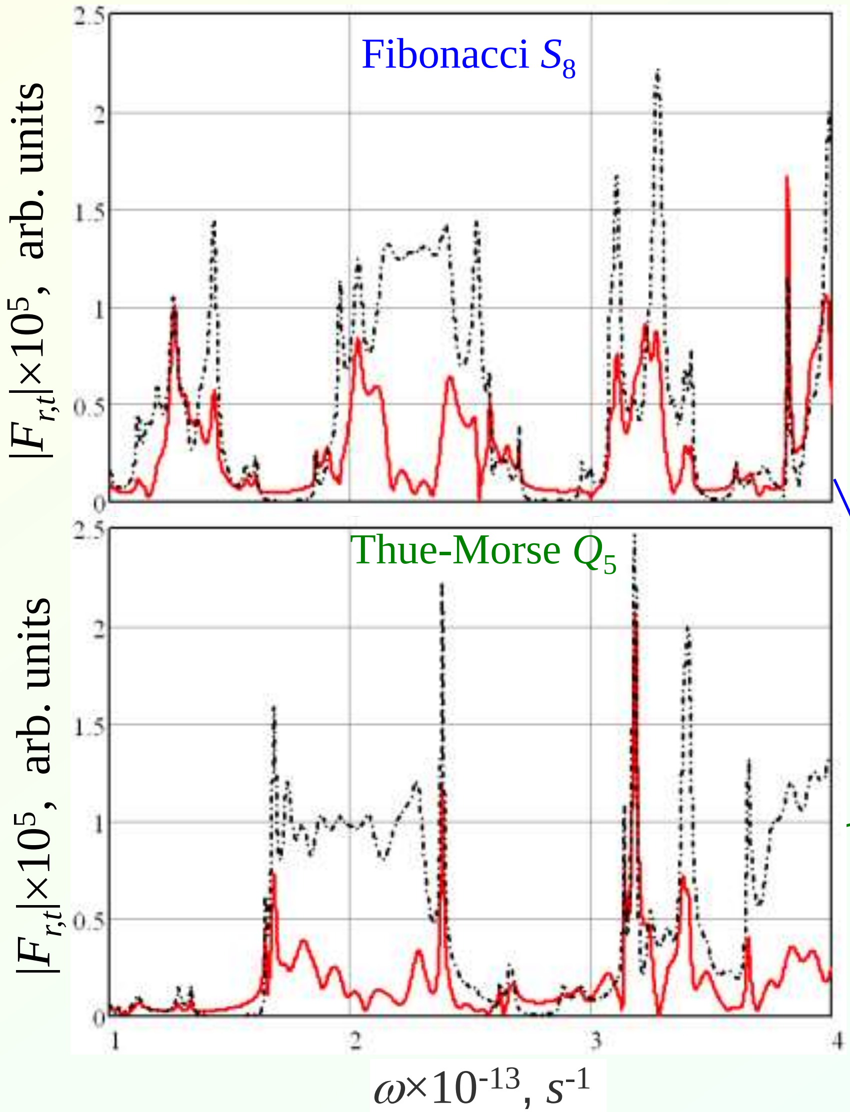
Thue-Morse Q_5 (32 layers, d_B=13µm)

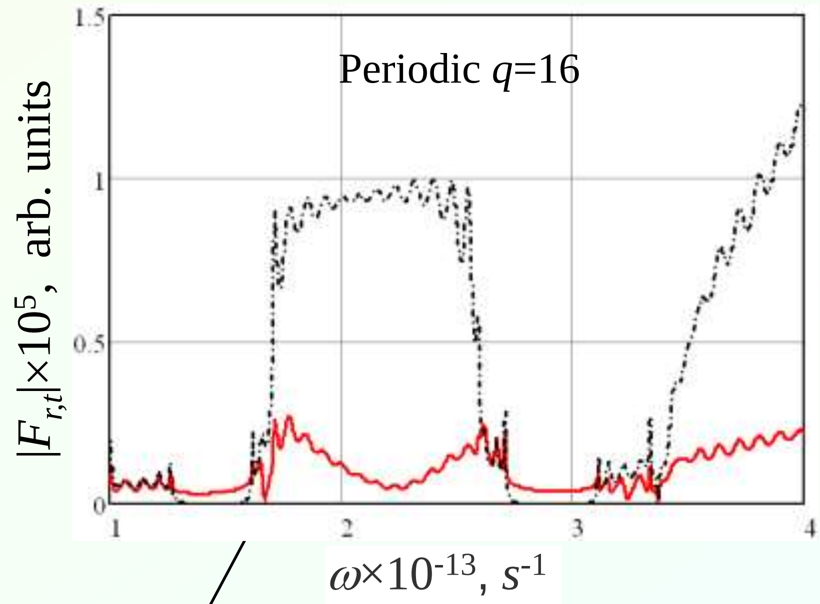
Θ_i = 30° (dashed lines)

Θ_i = 45° (solid lines)

$$d_A = d_B\left(1+\sqrt{5}\right)/2$$

ω_3 Emission: QPS vs Periodic





Periodic q=16 (32 layers, d_B=13μm)

Fibonacci S_8 (34 layers, d_B=12μm)

Thue-Morse Q_5 (32 layers, d_B=13μm)

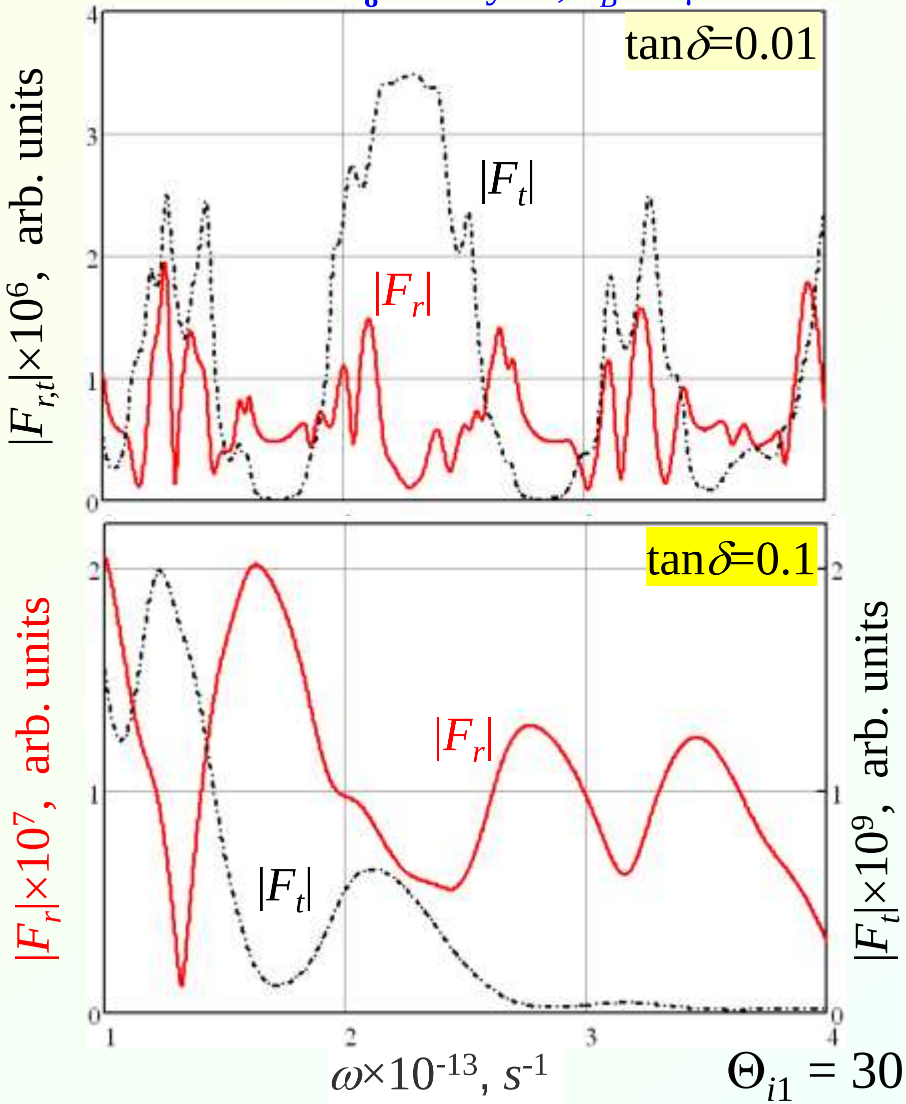
$\Theta_{i1} = 30°$, $\Theta_{i2} = 45°$; $d_A = d_B\left(1+\sqrt{5}\right)/2$

$|F_r|$ - solid lines

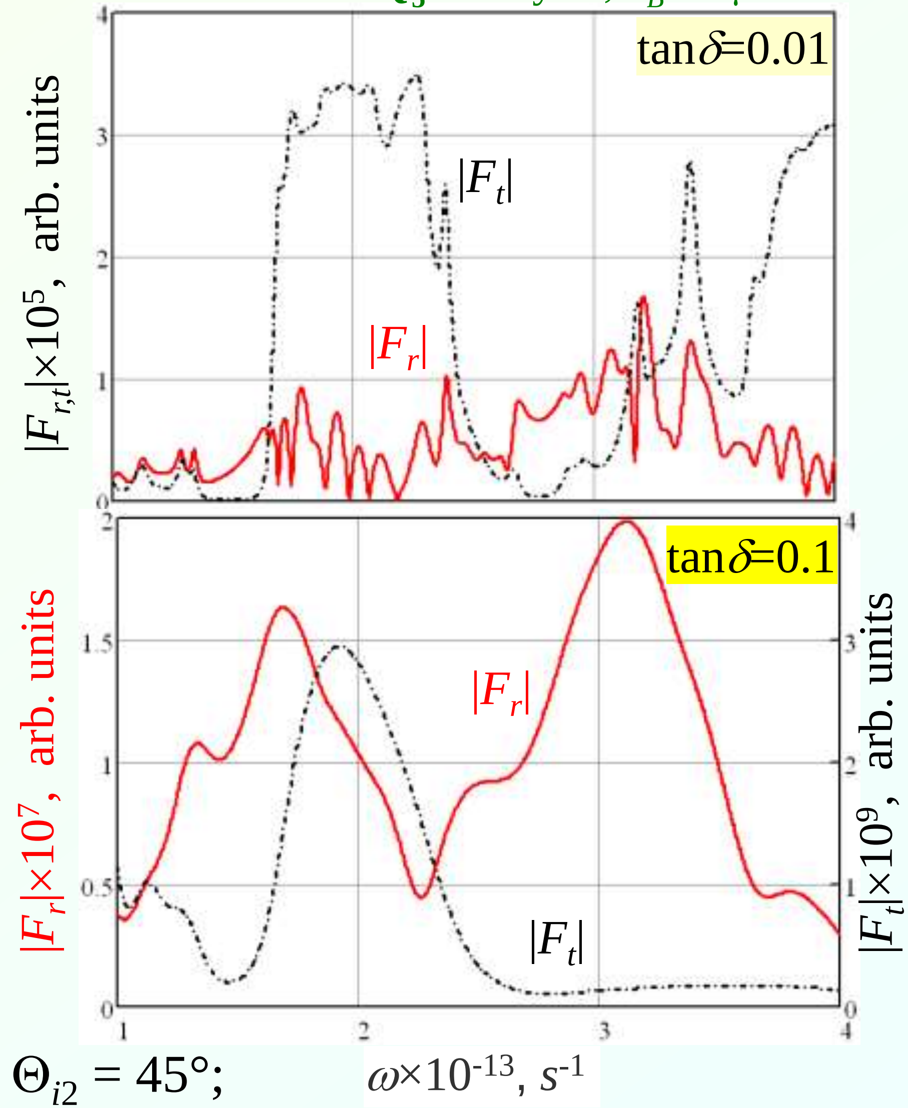
$|F_t|$ - dashed lines

Effect of Loss

Fibonacci S_8: 34 layers, d_B=12μm



Thue-Morse Q_5: 32 layers, d_B=13μm



$\Theta_{i1} = 30°$, $\Theta_{i2} = 45°$;

Concluding Remarks

- The semi-analytical approach to modelling combinatorial frequency generation by periodic and quasi-periodic multilayers has been developed
- The technique provides a unified framework for the analysis of periodic and quasi-periodic stacks
- It is shown that Fibonacci and Thue-Morse stacks can be treated like periodic structures with defects
- The developed theory is illustrated by simulations and provides an insight in the mechanisms of three-wave frequency mixing in the binary stacks.