Forum for Electromagnetic Research Methods and Application Technologies (FERMAT)

Recent Progress in “Slow” Methods for Computational Electromagnetics

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“Slow” methods = Controlled or “dialable” accuracy

- What do we mean by this?
  - Specify number of digits of accuracy in advance
  - Computer code reliably meets that specification (or tells user that it cannot)

- Byproduct: estimate of accuracy of results
  - Error bars

Controlled accuracy

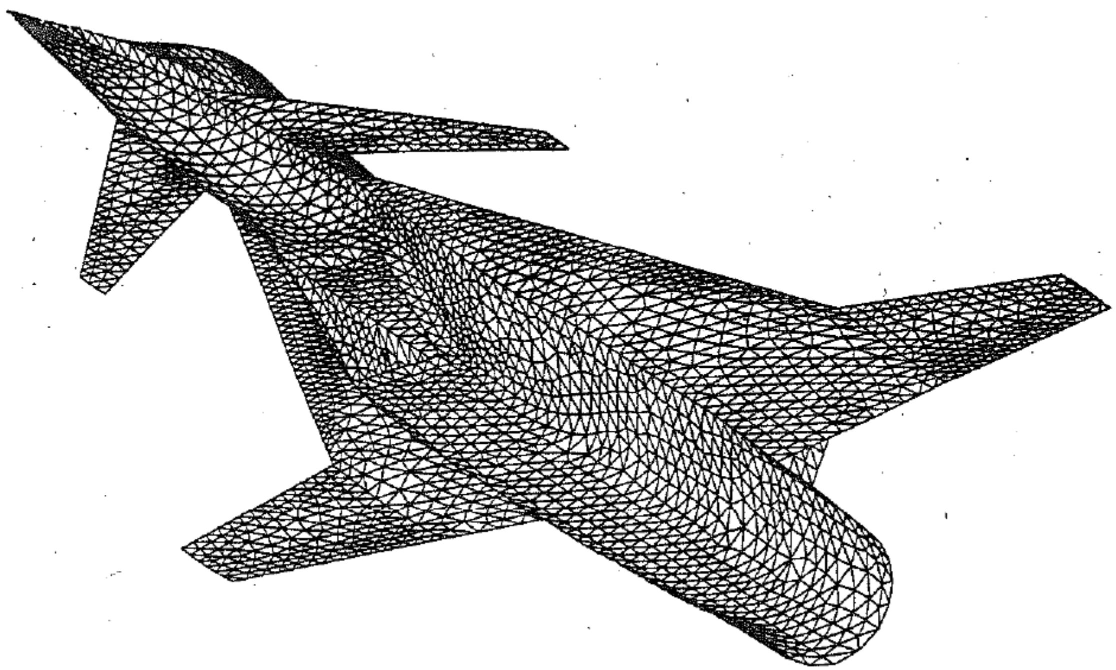
- Few applications demand “high” accuracy
  - Limited resolution of actual objects
  - Material parameters not well characterized
- Ability to obtain relatively high accuracy necessary for controlled accuracy
  - Controlled accuracy does not imply high accuracy
- Ultimately, controlled accuracy techniques should be the most efficient
  - They may be “slow” for now, but…

What are the necessary ingredients?

- Robust formulation
- High resolution model of object geometry/materials
- High order representations of currents/fields
- Treatment of edge/corner/tip singularities
- Accurate Green's function computations (IE)
- Means of estimating error
- Adaptive refinement (*h*-, *p*-, *hp*-) for efficiency

Typical practice today

- Techniques based on “low order” representations
  - Example: RWG basis functions, flat faceted model



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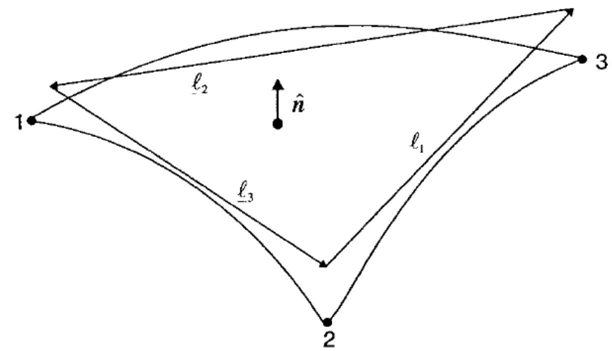
What's wrong with RWG?
Low-order accuracy

- RWG currents behave as O(h) or O(h^2) in the best case
- RWG Galerkin EFIE SCS behaves as O(h^3), a superconvergent rate, in the best case
- Low order methods make it difficult to improve accuracy very much, also difficult to estimate accuracy

Opposite of low order: High order

- Basis functions of polynomial degree *p*
  - Complete/mixed-order/interpolatory/hierarchical
- Geometry represented by curved cells
  - Exact/approximate
- Computational precision
  - Must be adapted as needed



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High order bases offer 2 advantages

- Better accuracy (but not necessarily "dialable")
- Reduced density of unknowns
  - RWG: 200 unknowns per sq. wavelength
  - WIPL: 30 unknowns per sq. wavelength
  - Recent paper by Ludick, Tonder, Jakobus (EMSS), *ICEAA* 2013:

Method:	RWG	HOBF
Time for matrix setup	1.9 hr	38.6 min
Time for matrix solution	7.3 hr	4.5 min
Memory usage (whole solution)	26.2 GB	1.2 GB

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How do we measure "error"?

- Error estimator used for research purposes:
  - Solve an over-determined system (2*N* by *N*)
  - Residual error available as byproduct:

$$NRE = \frac{\sqrt{\int \left| E_z^{inc} + E_z^s \right|^2 dt}}{\sqrt{\int \left| E_z^{inc} \right|^2 dt}} \cong \frac{\sqrt{\sum_{i=1}^{mq} w_i \left| E_z^{inc}(t_i) + E_z^s(t_i) \right|^2}}{\sqrt{\sum_{i=1}^{mq} w_i \left| E_z^{inc}(t_i) \right|^2}}$$

  - Good correlation with actual error in problems where we can compute the actual error

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How do we measure "error"?

– Residual error defined locally or globally

$$NRE = \frac{\sqrt{\int \left| E_z^{inc} + E_z^{s} \right|^2 dt}}{\sqrt{\int \left| E_z^{inc} \right|^2 dt}} \cong \frac{\sqrt{\sum_{i=1}^{mq} w_i \left| E_z^{inc}(t_i) + E_z^{s}(t_i) \right|^2}}{\sqrt{\sum_{i=1}^{mq} w_i \left| E_z^{inc}(t_i) \right|^2}}$$

– Slope of residual error curves often useful

$$Slope_q = \frac{\log_{10}(NRE_2) - \log_{10}(NRE_1)}{\log_{10}(N_2) - \log_{10}(N_1)}$$

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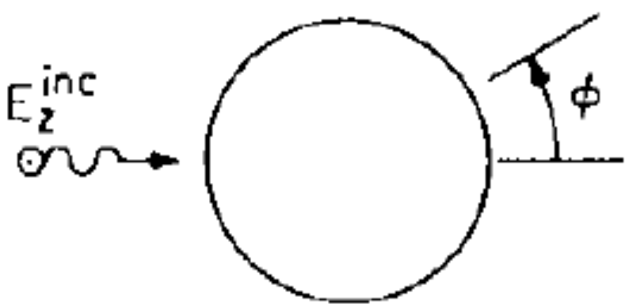
Simple (simplest?) example

- Circular PEC cylinder, radius a = 1 wavelength
  - TM-to-z MFIE
  - Exact circular model
  - Uniform plane wave excitation
  - MoM solution w/ q Legendre polynomial basis functions per cell and point testing at 2q locations



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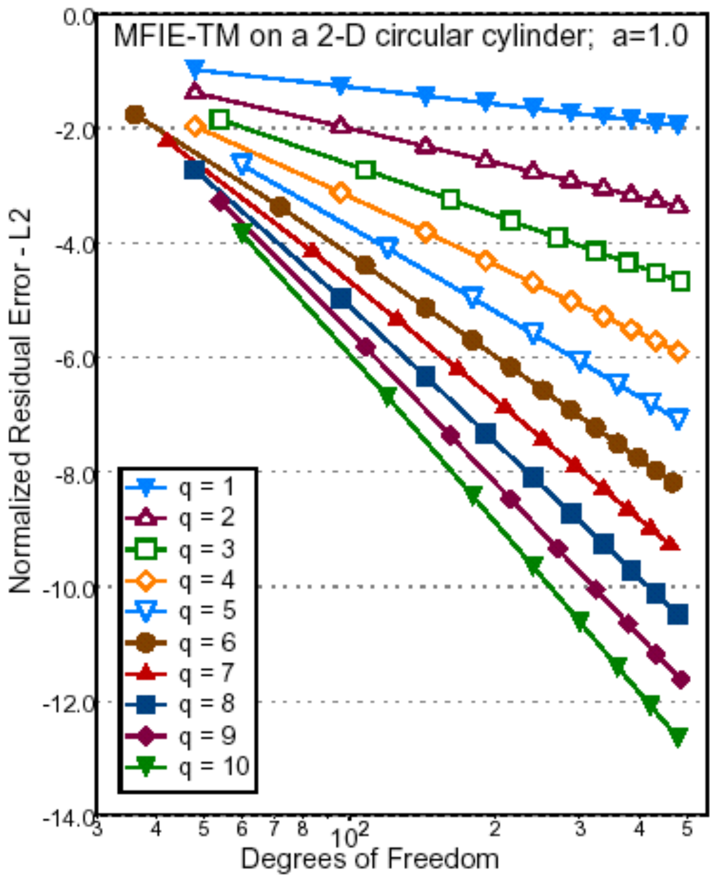
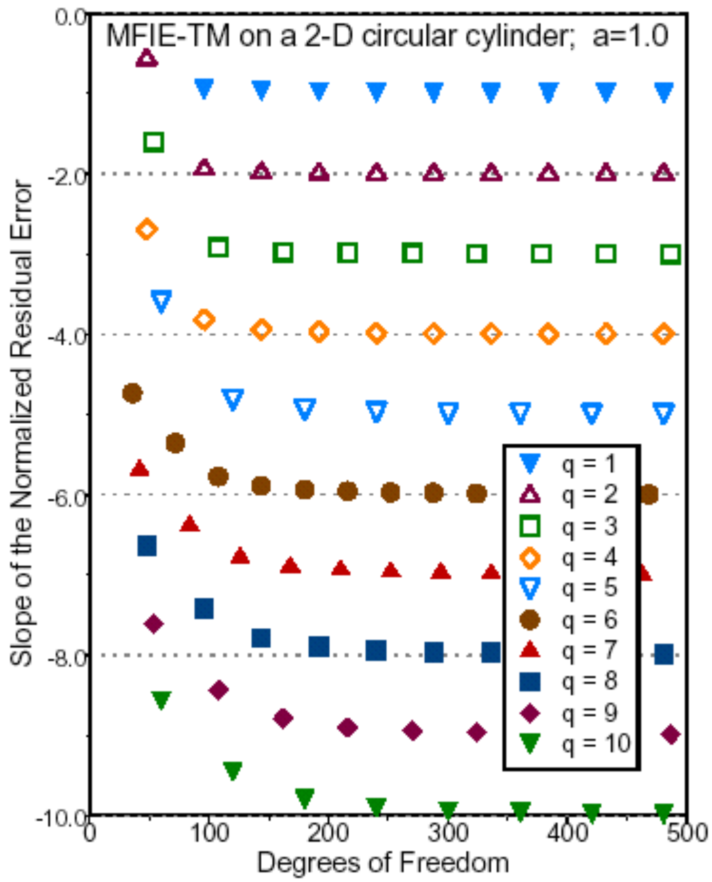
Example: Circular cylinder





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Observation

- When we see convergence rates (slopes) that are the same as the circular cylinder problem, have high confidence in the results
  - things are working OK!
- Reality: don't see this kind of convergence without doing (almost) everything right

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What we learned: smooth targets (*spheres*, *toroids*, *prolate spheroid*)

- High order techniques work well for smooth targets
  - Slopes of error curves increase with p
  - Observe consistent error slopes
  - More efficient use of computation
- Biggest challenge: maintaining high precision in Green's function calculations for integral eqns

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What we learned: linear dipole

- Need accurate Green’s function integrations
- Cannot reduce residual error uniformly without including charge singularities at discontinuities
  - special basis for singularity needed for end of hollow tube, flat end cap models
- Helps to have variable cell sizes
  - smaller cells where more rapid variation in excitation or current density encountered

Current/charge modeling at edges

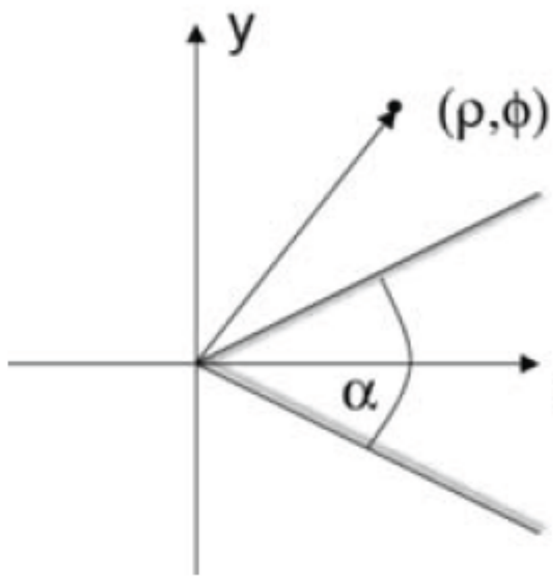
- Polynomial basis functions do not properly model the current and charge density near the edge
- Do not obtain high order behavior (steeper slopes)
- Local residual error does not decrease near surface discontinuity

High order approach for edges:

$$J_z \sim \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\rho^{2m+\upsilon_n-1}$$

$$J_\rho \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\rho^{2m+\upsilon_n}$$

$$\upsilon_n = \frac{n\pi}{(2\pi-\alpha)}$$



Consider *multiple* terms from the wedge solution in order to produce residuals with steeper slopes

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Example: 60 degree corner

Table I.
Example: Exponents of the degrees of freedom used for a given order, for cells near a 60 degree corner (TM case). The corner cells involve twice as many degrees of freedom as the other cells, and are twice as large.

Order of representation q	Exponents used in non-corner cells	Exponents used in corner cells
1	0	–2/5, 0
2	0, 1	–2/5, 0, 1/5, 1
3	0, 1, 2	–2/5, 0, 1/5, 1, 4/5, 2
4	0, 1, 2, 3	–2/5, 0, 1/5, 1, 4/5, 2, 7/5, 3
5	0, 1, 2, 3, 4	–2/5, 0, 1/5, 1, 4/5, 2, 7/5, 3, 8/5, 4

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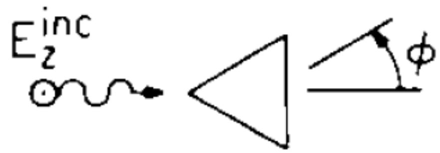
Triangular cylinder target

- TM PEC cylinder, equilateral triangle cross section
  - 7 wavelengths per side
  - Corner-on incident uniform plane wave
  - MFIE



- Expansions of order $q = 2, 3, 4$, with q unknowns (degrees of freedom) per non-corner cell

- Add q singular terms in corner cells

- Uniform cell sizes except corner cells twice as large

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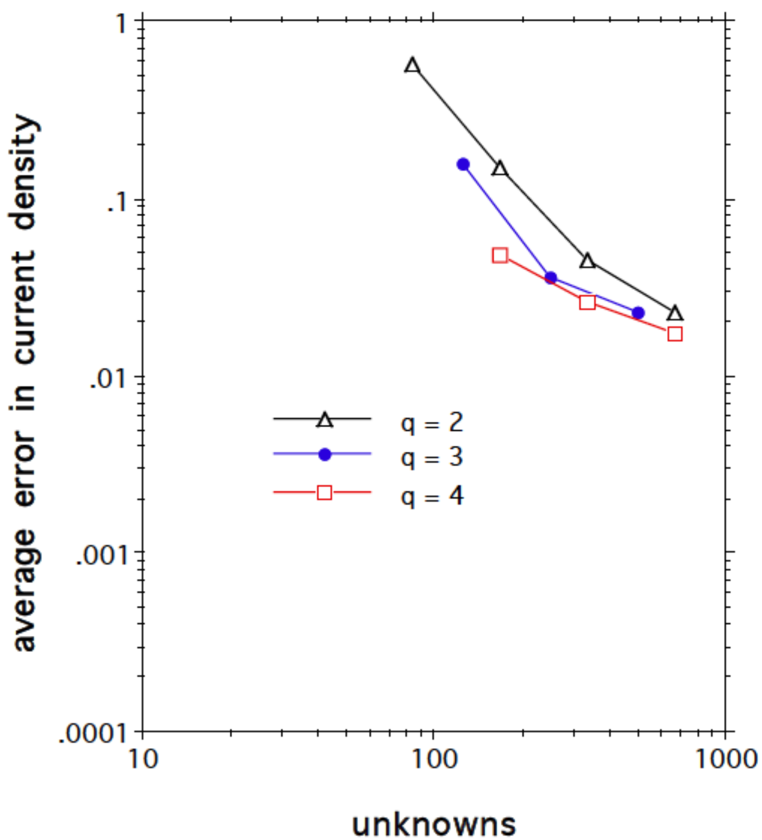
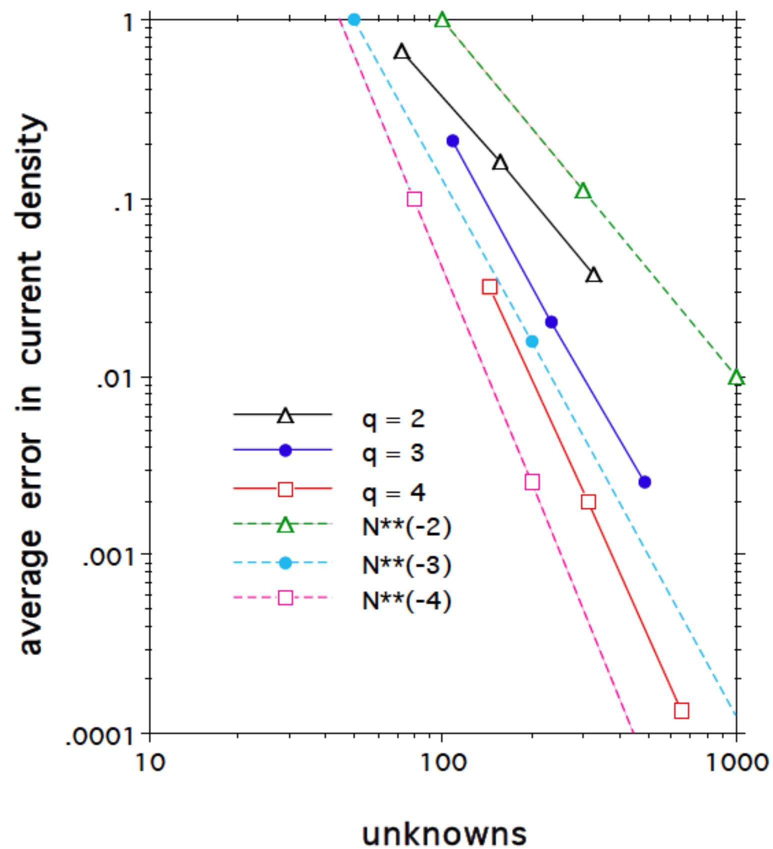
Polynomial vs. Singular expansions





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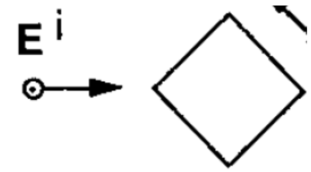
→ Actual error (NOT residual error)

Square Cylinder

- TM PEC cylinder, square cross section
  - 7 wavelengths per side
  - Corner-on incident uniform plane wave
  - MFIE solution

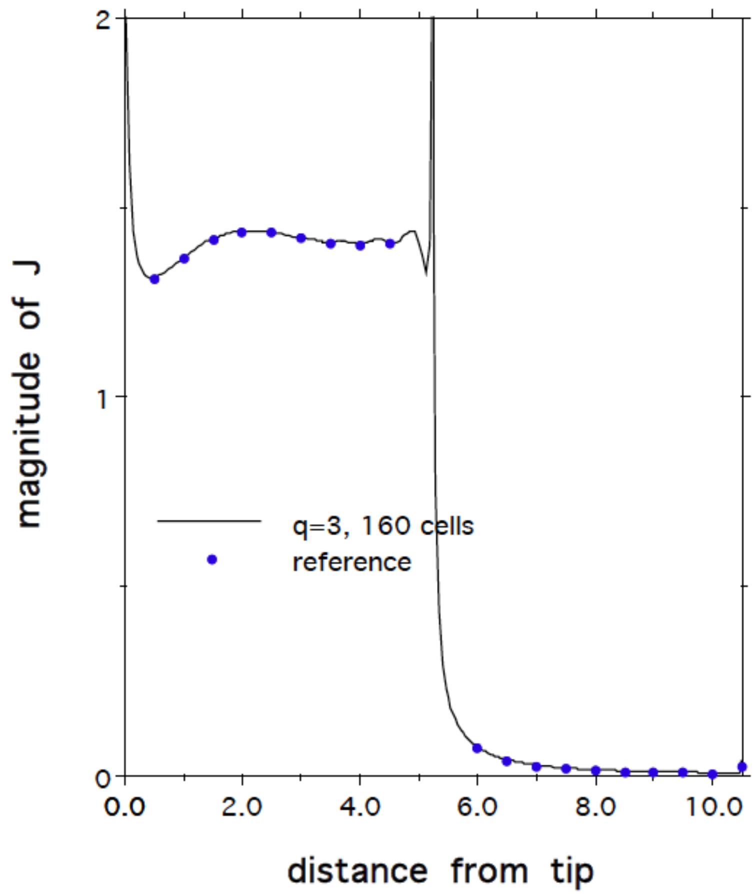


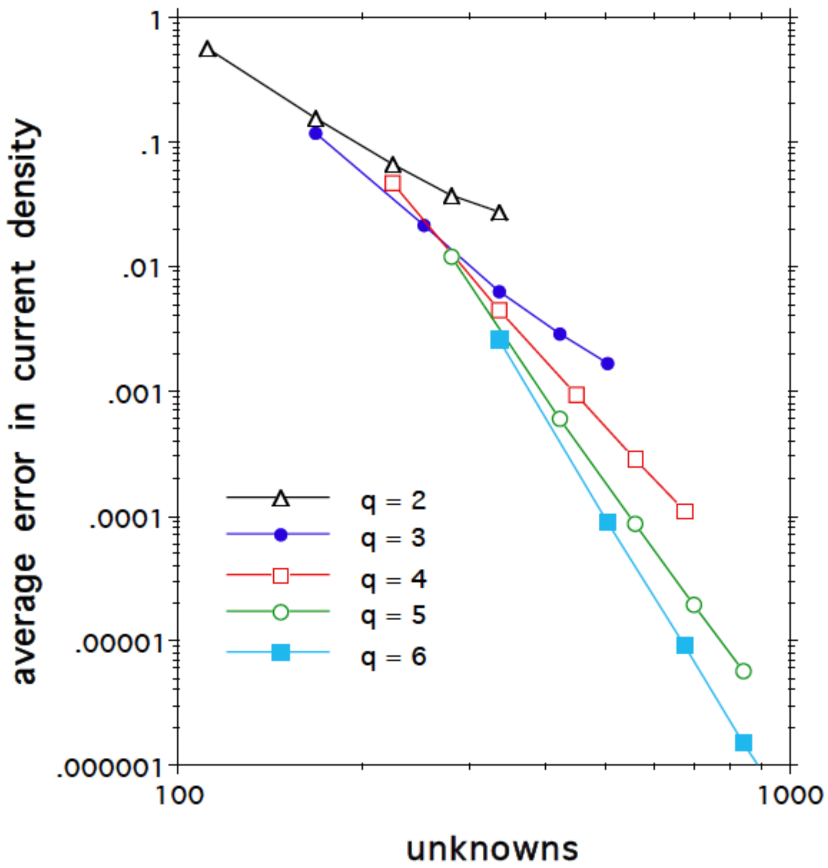
- Employ expansions of order $q = 2, \ldots, 6$, with q unknowns (degrees of freedom) per non-corner cell

- Add q singular terms in corner cells

- Uniform cell sizes except corner cells twice as large

Square Cylinder





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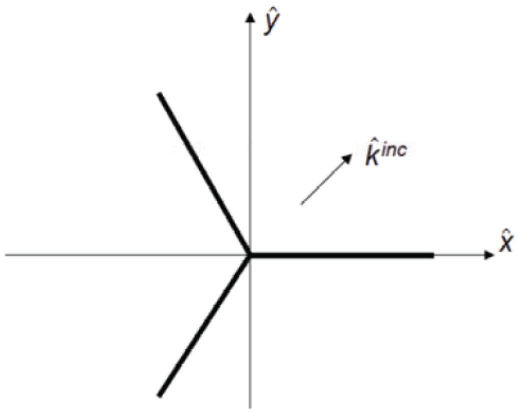
Observation

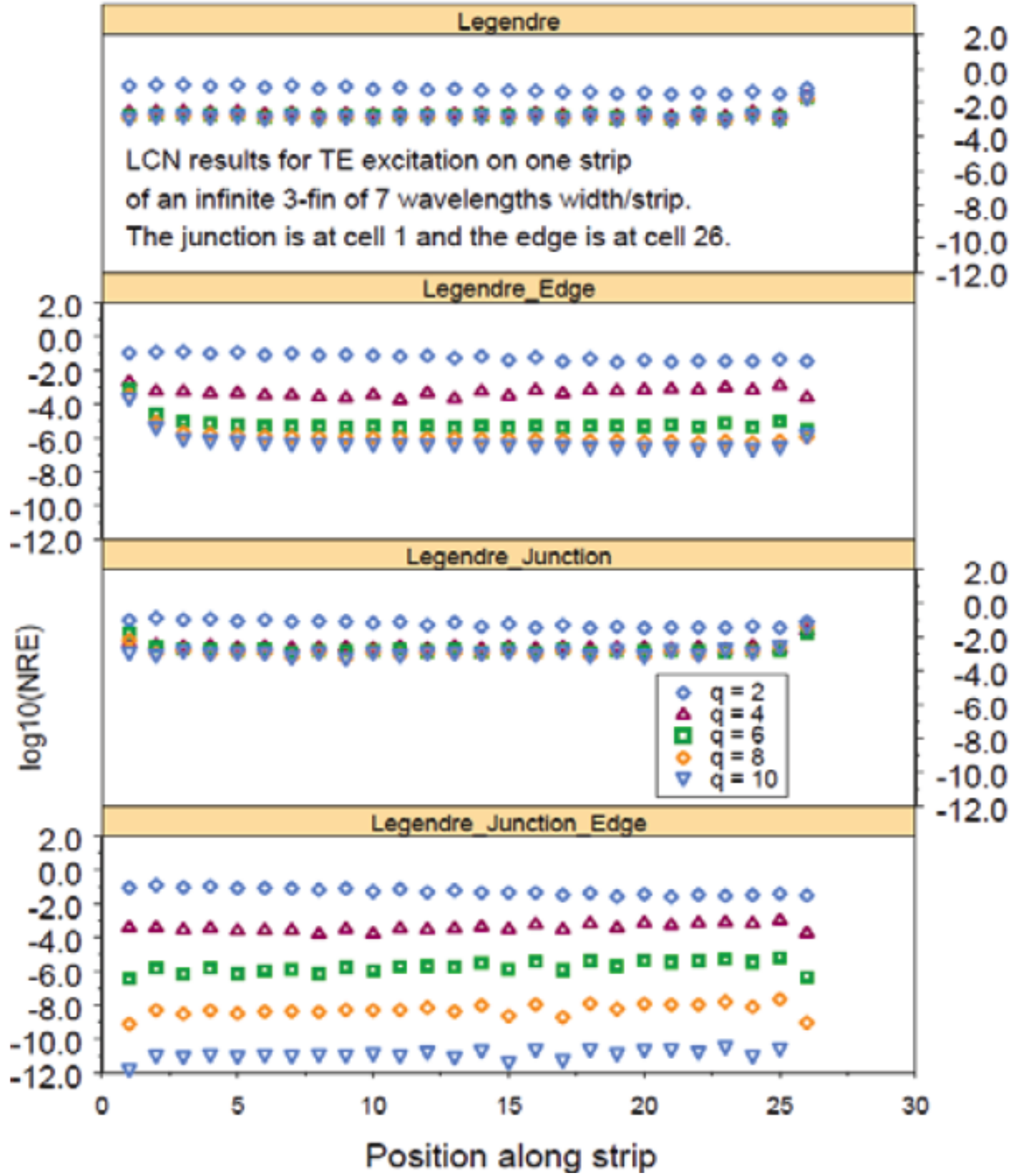
- Convergence rates (slopes) for triangle and square cylinder are the same as the circular cylinder
  - true high order behavior
  - things are working OK!

- Current density for $q = 7$ has converged to at least 5 decimal places

3-fin structure



- local NRE for:
  - poly only
  - poly + end
  - poly + junction
  - poly + end + junction



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Recommendation (2D):

- Starting point for analysis of problems with edges/corners:
  - Use uniform cell sizes, except that corner cells are twice the size of other cells
  - Use uniform order q in non-corner cells
  - Use an additional q singular terms in corner cells
- h-refinement should yield high order behavior
- Ultimately, employ some form of adaptive h/p refinement

Tip/Corner Singularities in 3D

- Vertex of plane angular sector has exponents associated with Lame functions
  - corners of plate and aperture
  - (Satterwhite & Kouyoumjian, 1970)
- Tip of circular cone has exponents associated with the zeros of Legendre functions
  - (Van Bladel, various)
- Tip of elliptic cone has exponents associated with Lame functions
  - (Boersma & Jansen, 1990)

3D target #1: rectangular plate

- Think we know how to handle plate edges based on the approach that works in 2D
  - Adapt preceding approach to a surface, determine how much larger to make edge cells to balance local residual errors, etc.
- Need to investigate plate corners
  - Can we reduce residual errors without modeling the corner singularity?
- Incorporate corner behavior into special basis functions
  - Goal is to produce residual errors that exhibit same slopes as circular cylinder problem

Rectangular plate corner current

$$J_r \sim \frac{1}{\sin\phi}\sum\left\{\begin{array}{l} A_{-1}(\phi)r^{\nu_{oi}-1} + A_0(\phi)r^{\nu_{ei}} + A_1(\phi)r^{\nu_{oi}+1} \\ +A_2(\phi)r^{\nu_{ei}+2} + A_3(\phi)r^{\nu_{oi}+3} + \ldots \end{array}\right\}$$

$$J_\phi \sim \sin\phi\sum\left\{B_{-1}(\phi)r^{\nu_{oi}-1} + B_1(\phi)r^{\nu_{oi}+1} + B_3(\phi)r^{\nu_{oi}+3} + \ldots\right\}$$

$$\nabla_S \bullet \bar{J} \sim \frac{1}{\sin\phi}\sum\left\{\begin{array}{l} C_{-1}(\phi)r^{\nu_{ei}-1} + C_0(\phi)r^{\nu_{oi}} \\ +C_1(\phi)r^{\nu_{ei}+1} + C_2(\phi)r^{\nu_{oi}+2} + \ldots \end{array}\right\}$$

index	ν_{oi}	ν_{ei}
1	0.81466	0.29658
2	1.59713	1.13125
3	1.95533	1.42651
4	2.52088	2.00000

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Conclusions

- Today, a wide range of 2D problems can be solved to high accuracy even when edge singularities are present
  - have yet to demonstrate “controlled” accuracy with integral equations, but we are close
  - need reliable & efficient error estimators, adaptive refinement schemes

- Still have quite a way to go to achieve “controlled” accuracy for general 3D problems
  - missing pieces: tip singularities, error estimators, better grasp of model error impact, …