A Look at Electromagnetic Field Transformation using Transformation Optics (TO), Wave Equation and Scattering Matrix Formalisms

Raj Mittra\textsuperscript{1,2,*}, Yuda Zhou\textsuperscript{3} and Pengfei Zhang\textsuperscript{1,4}

\textsuperscript{1}Department of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816 USA
\textsuperscript{2}Department of Electrical and Computer Engineering, King Abdulaziz University, Jeddah 22254, Saudi Arabia
\textsuperscript{3}Department of Electrical Engineering, The Pennsylvania State University, University Park, PA 16802 USA
\textsuperscript{4}School of Electrical Engineering, Xidian University, Shaanxi 710071 China

\textsuperscript{*}email: rajmittra@ieee.org

Abstract—The main objective of this paper is to take a look at Electromagnetic Field Transformation from the Wave Equation point of view, which is slightly different from the traditional one based on the Transformation Optics (TO) approach, as we explain below. We begin with a review of Transformation Optics (TO), which provides a novel paradigm for designing a whole host of electromagnetic devices by using geometry transformation. The underlying concepts of TO as well as its theoretical foundations are well established, thanks to the pioneering works of a number of leading researchers, which date back to 2006. In this paper we examine the realizability issue of the materials, prescribed by the TO, from a practical point of view and identify some difficulties we may encounter in the process of fabricating electromagnetic devices such as cloaks and microwave antennas following the TO prescription. Next, we introduce a Wave-equation-based (WEB) approach to field transformation to examine whether we could mitigate the problems encountered with the TO that we mentioned above. Finally, we introduce the Generalized Scattering Matrix (GSM) concepts of Field Transformation, and we use it to modify the performance metrics of the devices in order to render them realizable by using available materials, without the need to use Metamaterials that are often lossy, narrowband, and polarization-dependent.

Index Terms—Electromagnetic Field Transformation; Transformation Optics (TO); Wave Equation; Scattering Matrix Formalisms

I. INTRODUCTION

Perhaps no other topic has caught the recent attention of physicists and microwave engineers than the subject of Transformation Optics (TO) aka Transformation Electromagnetics (TEM), as the number of publications in the scientific literature [1-25] would indicate. This is because the TO provides an elegant theoretical approach for designing a variety of electromagnetic devices, e.g., cloaks, flat lenses, reflectarrays, to name just a few, based on a fresh new way of looking at the problem that relates the electromagnetic fields in one coordinate system to another. For instance if we wish to design a flat lens following the TO algorithm, we can start with a plano-convex lens in free space and transform it into a rectangular shape with material parameters dictated by the TO, which maps the electromagnetic fields from one space to the other so that the two sets of fields are equal, and the performance of the flat lens exactly mimics that of the convex lens, provided we can realize the requisite material parameters of the flat lens, and of the medium surrounding the lens, in accordance with the dictates of the TO algorithm. Since the TO paradigm is well researched, and the algorithm itself is based on solid theoretical foundations of co-ordinate transformation of Maxwell’s equations [1, 2], it is legitimate to wonder if we really need another contribution on the subject, unless it adds something new and not just rehashes what is well known already. Our objective in this paper is indeed to introduce a different way of looking at the TO approach, focusing more on the practical aspect of realizing the electromagnetic devices whose geometries we choose, and whose performance we desire to emulate such that it corresponds to that of an existing device with a different geometrical shape. We have underscored the word “performance”
because, unlike TO, we may choose to work with certain selected “performance metrics” of the devices rather than attempting to render the Electric and Magnetic field distributions of the original and transformed devices to be strictly identical, as we do when we follow the TO algorithm. To follow our chosen path, we also deviate from the TO strategy, which works with Maxwell’s equations for both the Electric and Magnetic fields, and transforms these field equations from one space to another by using the rules of coordinate transformation to determine the material parameters of the geometrically-transformed device. Instead, we find that it is expedient for us to work with the “wave equation” satisfied by either the Electric or the Magnetic field, though we prefer the E-field wave equation because it enables us to work with the permittivity parameter \( \varepsilon \), and leave the permeability \( \mu \) unchanged. Working with \( \varepsilon \) alone, if possible, would be significantly advantageous from a practical point of view, since realizing magnetic materials typically called for by the TO [3] has remained an elusive problem, even when the material is isotropic (anisotropic \( \mu \) simply exacerbates the problem). Turning to Metamaterials (MTMs) does not really circumvent the realization problem, since the required MTMs can be lossy, narrowband, dispersive and highly polarization-dependent [43-48].

Following the discussion of the wave-equation-based (WEB) approaches, one of which is based on conformal mapping via the Schwarz-Christoffel transformation applied to the wave equation, we will turn to a slightly different formulation of the problem based on the Generalized Scattering Matrix (GSM) approach [26], and show how we can choose the performance metric of the electromagnetic device we wish to design to be slightly different from that of the TO, so that we can work with either available or with conveniently realizable materials, as opposed to MTMs typically called for by the TO. We will include a number of practical examples to illustrate some practical applications of the techniques we describe in this paper, reminding the reader that the practical aspects are indeed the main focus of our work.

II. Review of TO and Introduction of Wave-equation-based (WEB) approach

In this section we present the wave-equation-based (WEB) approach to field transformation, where our present goal—to be modified later in Sec.4 and explained why—is to preserve the \( E \) and \( H \) fields under coordinate transformation. Traditionally, TO accomplishes the field mapping by using the transformation of Maxwell’s equations from unprimed to prime coordinate systems as follows.

\[
\begin{align*}
\nabla' \times E' &= -j\omega \mu' H' \quad (1a) \\
\nabla' \times H' &= +j\omega \varepsilon' E' \quad (1b)
\end{align*}
\]

and shows that if we use the material parameters given by

\[
\begin{align*}
\varepsilon' &= \frac{\Delta\varepsilon\Lambda^T}{\det(\Lambda)} \quad (2a) \\
\mu' &= \frac{\Delta\mu\Lambda^T}{\det(\Lambda)} \quad (2b)
\end{align*}
\]

where

\[
\Lambda = \begin{bmatrix}
\frac{\partial \xi'_1}{\partial \xi_1} & \frac{\partial \xi'_1}{\partial \xi_2} & \frac{\partial \xi'_1}{\partial \xi_3} \\
\frac{\partial \xi'_2}{\partial \xi_1} & \frac{\partial \xi'_2}{\partial \xi_2} & \frac{\partial \xi'_2}{\partial \xi_3} \\
\frac{\partial \xi'_3}{\partial \xi_1} & \frac{\partial \xi'_3}{\partial \xi_2} & \frac{\partial \xi'_3}{\partial \xi_3}
\end{bmatrix}
\]

then the \( E \)- and \( H \)-fields map one-to-one from one coordinate system to another, and we can achieve the same performance from the transformed device as we did from the original one, whose geometry we transformed from the original to a more desirable one. Experience has shown that the realization of the TO-dictated materials, examples of which are shown in Fig.1 for the relatively simple geometry of a cylindrical cloak, is a daunting task, which is yet to be tackled satisfactorily even for such a simple geometry, let alone for a complex structure such as an airplane or a missile.

Before we discuss the WEB approach, we turn to examine another fundamental limitation posed by the TO paradigm when we attempt to reduce the thickness of the cloak \( t = R_2 - R_1 \) to realistic values, \( e.g. \), a small fraction of the operating wavelength. We note that the material values \((\varepsilon, \mu)\) do not change near the outer boundary of the cloak, located at \( \rho = R_2 \), regardless of
whether the thickness $t$ of the cloak is large or small; and, for that matter, neither do the behaviors of $(\varepsilon, \mu)$ of the cloak in the neighborhood of $\rho = R_1$, where the above parameters approach extreme values. While we can attempt to partially mitigate this problem by imposing a chosen limit on the values of these parameters, we cannot change the fact that the relative $(\varepsilon(\rho), \mu(\rho))$ must reduce from extreme values at the surface of the cylinder to moderate ones within a relatively short distance, which poses realizability problems in practice. Practical realizability of thin invisibility cloaks have not met with too much success in the past for this reason, and it is unlikely that the thickness issue will be resolved anytime soon if we continue to impose the above invisibility criterion on the TO-based cloak designs. Fig.2 shows the deterioration of the performance of the cloak (see Fig.2(c)) when we design a thin three-layer cloak by using discretized values of the continuous cloak parameters. It becomes evident, when we compare the performance of the fabricated thin cloak with the ideal one, that the wavefront of the field becomes considerably distorted when we use a thin cloak. We will address this problem later when we introduce a different performance metric for the cloak (or blanket) in Sec.3, and discuss a way we can use a thin coating to reduce the scattering form a target, albeit in the backscattering hemisphere.

For the WEB approaches, which we present below, and which deal with coordinate transformation of the wave equation, we will initially limit ourselves to two-dimensional (2D) problems, deferring to examine the more complex 3D case until later. We can gain much insight from the study of the 2D cases and draw some conclusions that are also valid and applicable to the three-dimensional case. We hasten to point out that similar two-dimensional cases have also been studied extensively by other workers, both theoretically and experimentally, and the cylindrical and carpet cloaks are but two examples of the 2D problem that have been researched earlier [3, 4].

Taking a cue from the literature we investigate the cases of conventional and carpet cloak designs, for an infinite cylinder and a triangular bump in an infinite PEC plane, respectively. We assume, also in common with the existing literature pertaining
to these geometries, that the fields are independent of the longitudinal direction \( z \), and that all the fields can be derived from the \( E_z \)-component of the Electric field, which is the only non-zero component of the Electric field, from Maxwell's equations

\[
\nabla \times \mathbf{E}(x,y) = -j \omega \mu \mathbf{H}(x,y) \tag{4a}
\]

\[
\nabla \times \mathbf{H}(x,y) = +j \omega \varepsilon \mathbf{E}(x,y) \tag{4b}
\]

by setting \( \mathbf{E} = 2E_z \). This case has been referred to in the physics literature [5] as the TE\(_z\) case, although in the conventional waveguide terminology [27, 28] this case is termed TM\(_z\) instead, and we will follow this latter terminology in this work.

At this point we set \( \mu = \mu_0 \) at the outset, and we will soon see how, in contrast to the TO paradigm, the WEB approach enables us to do this in a straightforward way. As we well know, finding materials with \( \mu \) values called for by the TO has been a severe roadblock and we wish to avoid this problem altogether, or to rely on MTMs for their realization. We wish to see what we can do instead with \( \varepsilon \)-only materials for the cloaks and study how the WEB approach provides a very simple way to do this. Furthermore, we assume that \( \varepsilon \) is isotropic (for 2D problems), and that it is invariant of the longitudinal coordinate \( z \). If successful, this would obviously facilitate the realization of the cloak, since anisotropic materials with arbitrary properties, typically dictated by the TO, are very difficult to realize in practice.

Next, we return the two Maxwell’s curl equations and eliminate \( \mathbf{H} \), by using

\[
\mathbf{H}(x,y) = -\frac{1}{j\omega \mu_0} \nabla \times \mathbf{E}(x,y) \tag{5}
\]

and substituting it into (4b), to get

\[
\nabla \times (\nabla \times \mathbf{E}(x,y)) - \omega^2 \mu_0 \varepsilon \mathbf{E}(x,y) = 0 \tag{6}
\]

Using vector identities, we rewrite (6) as

\[
\nabla (\nabla \cdot \mathbf{E}(x,y)) - \nabla^2 \mathbf{E}(x,y) - \omega^2 \mu_0 \varepsilon \mathbf{E}(x,y) = 0
\]

(7)

From Maxwell's Equations, in a source-free region, we also have

\[
\nabla \cdot \mathbf{D}(x,y) = \nabla \cdot [\varepsilon \mathbf{E}(x,y)] = \nabla \varepsilon \cdot \mathbf{E}(x,y) + \\
\varepsilon \nabla \cdot \mathbf{E}(x,y) = \varepsilon \nabla \cdot \mathbf{E}(x,y) = 0 \tag{8}
\]

which enables us to simplify (7) to:

\[
(\nabla^2 + k^2)\Psi = 0 \tag{9}
\]

which is the Helmholtz equation (wave equation) satisfied by \( \Psi = E_z \). We wish to see how this equation transforms from one coordinate system to another, so that the large cylinder covered by an appropriately designed cloak behaves similar to the small one surrounded by free space instead.

Thus we begin with the problem of designing a cloak to reduce the scattering from a circular cylinder using the WEB approach by looking at a transformation of the two coordinate systems that relates the first system, namely the cylindrical region of \( 0 < \rho < b \) to the second, which is an annular region \( a < \rho' < b \), where \( \rho \) and \( \rho' \) are the radial coordinates of the two systems, respectively; \( a \) is the radius of the cylinder; and \( b \) is the outer radius of the cloak region, as shown in Fig.3 (also see [3], where \( R_1 \) and \( R_2 \) correspond to \( a \) and \( b \), respectively). A linear transformation that relates the two coordinates is

\[
\rho' = \frac{b-a}{b} \rho + a; \quad \phi' = \phi; \quad z' = z \tag{10}
\]

where \( \phi \) and \( z \), respectively, are the angular and vertical coordinates in the original system (which is inherently free space) and \( \phi' \) and \( z' \) are the corresponding angular and vertical coordinates in the transformed system, respectively, which includes the cylinder and the cloak.

As mentioned earlier, we wish to look at the problem from the perspective of wave equations, more specifically, the Helmholtz equation satisfied by the \( E_z \)-field:

\[
\nabla^2 E_z + k_0^2 E_z = 0 \tag{11}
\]

For the cylindrical system under consideration here, we rewrite (11) as...
\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z(\rho, \phi)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z(\rho, \phi)}{\partial \phi^2} + k_0^2 E_z(\rho, \phi) = 0
\]  
(12)

or simply,
\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z(\rho, \phi)}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z(\rho, \phi)}{\partial \phi^2} + \omega^2 \mu_0 \varepsilon_0 E_z(\rho, \phi) = 0
\]  
(13)

In the transformed coordinate \( \rho', \phi, z \) system, the Helmholtz equation of the \( E \)-field can be derived to read as follows
\[
\frac{1}{\rho' - a} \frac{\partial}{\partial \rho'} \left[ (\rho' - a) \frac{\partial E_z(\rho', \phi)}{\partial \rho'} \right] + \frac{1}{(\rho' - a)^2} \frac{\partial^2 E_z(\rho', \phi)}{\partial \phi^2} + k_{0'c}^2 E_z(\rho', \phi) = 0
\]  
(14)

and since we have chosen the permeability \( \mu \) to be that of free space, i.e., \( \mu_0 \) (14) can be written in the form
\[
\frac{1}{\rho' - a} \frac{\partial}{\partial \rho'} \left[ (\rho' - a) \frac{\partial E_z(\rho', \phi)}{\partial \rho'} \right] + \frac{1}{(\rho' - a)^2} \frac{\partial^2 E_z(\rho', \phi)}{\partial \phi^2} + \omega^2 \mu_0 \varepsilon_0 \left( b - a \right)^2 E_z(\rho', \phi) = 0
\]  
(15)

where the cylinder of radius \( a \) is mapped to a cylinder of limiting radius (tending to zero). In the general case, we denote \( c \) to be the radius of the small cylinder as shown in Fig.4, which yields
\[
\rho' = \frac{b-a}{b-c} \rho + b \frac{a-c}{b-c}; \quad \phi' = \phi; \quad z' = z
\]  
(16)

For this scenario the Helmholtz equation satisfied by the \( E \)-field can be rewritten as
\[
\frac{1}{\rho' - b \frac{a-c}{b-c}} \frac{\partial}{\partial \rho'} \left[ \left( \rho' - b \frac{a-c}{b-c} \right) \frac{\partial E_z(\rho', \phi)}{\partial \rho'} \right] + \frac{1}{(\rho' - b \frac{a-c}{b-c})^2} \frac{\partial^2 E_z(\rho', \phi)}{\partial \phi^2} + \omega^2 \mu_0 \varepsilon_0 \left( b - a \right)^2 E_z(\rho', \phi) = 0
\]  
(17)

Next we shift the origin of the \( \rho', \phi, z \) system by defining
\[
\rho_1 = \rho' - b \frac{a-c}{b-c}
\]  
(18)

and rewrite (17) as
\[
\frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \left[ \rho_1 \frac{\partial E_z(\rho_1 + b \frac{a-c}{b-c}, \phi)}{\partial \rho_1} \right] + \frac{1}{\rho_1^2} \frac{\partial^2 E_z(\rho_1 + b \frac{a-c}{b-c}, \phi)}{\partial \phi^2} + \omega^2 \mu_0 \varepsilon_0 \left( b - a \right)^2 E_z(\rho_1 + b \frac{a-c}{b-c}, \phi) = 0
\]  
(19)

In the \( \rho' \) or \( \rho_1 \) system, the medium parameter are \( \mu_0 \) and \( \varepsilon_r \varepsilon_0 \), where \( \varepsilon_r \) is yet to be determined. Let
\[
k_1^2 = k_0^2 \varepsilon_r = \frac{k_0^2}{b - a}
\]  
(20)

which yields
\[
\varepsilon_r = \left( \frac{b-c}{b-a} \right)^2
\]  
(21)

Then \( E_z \) satisfies:
\[
\frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \left[ \rho_1 \frac{\partial E_z(\rho_1 + b \frac{a-c}{b-c}, \phi)}{\partial \rho_1} \right] + \frac{1}{\rho_1^2} \frac{\partial^2 E_z(\rho_1 + b \frac{a-c}{b-c}, \phi)}{\partial \phi^2} + k_1^2 E_z(\rho_1 + b \frac{a-c}{b-c}, \phi) = 0
\]  
(22)

which is the form of the wave equation in the \( \rho_1 \) system. The choice of the material parameters,
given in (21), renders the form of the two wave equations (13) and (22) in these systems the same, and enables us to achieve our goal.

For the limiting case $c \to 0$, we need to set

$$\epsilon = \left(\frac{b}{b-a}\right)^2 \epsilon_0$$

(23)

which is the case of the ideal cloak with no scattering. For reduced scattering we choose a non-zero but small $c$.

If we choose the outer radius $b$ of the cloak to be $2a$, and if we let $c = 0$, then we see from (23) that the $\epsilon_r$ value in the $\rho', \phi, z$ system would be $\epsilon_r = 4$. It is interesting to note, by referring Fig.1(c), that this is also the material parameter for $\epsilon_z$ derived by using the TO and presented in [5], where $\mu_\phi$ is seen to equal 1, just as the $\mu$ values we have chosen in our case to avoid realizability problems. However, $\mu$ is anisotropic in the TO case and $\mu_r$ is less than unity throughout the cloak region, tending to 0 at the inner boundary, which makes it difficult if not impossible to realize in practice. We will have further comments on this issue below, after we discuss the case of general transformation, which is non-linear in general.

One can see from (20), which is valid when the transform from the free space to the transformed space is linear, that the relative permittivity in the latter space is proportional to the square of the ratio of the areas of the original to the transformed cells. If we follow this paradigm for the non-linear transform, we can obtain the parameter values in the transformed space by using a locally-linear approximation. As an example, we consider the geometry in Fig. (5), where the region of $c < \rho < b$ is compressed into a region of $c < \rho' < b$ via a non-linear transform, with the cell sizes varying exponentially in the radial direction from small to large in the transformed space, such that the cell size at its outer boundary is identical to that in free space.

Let $\Delta\rho$ be the cell size (uniform) in the original (free space) system in $\rho$-direction and let $N$, the number of rings in the original system, be given by

$$N = \frac{(b - c)}{\Delta\rho}$$

(24)

In the transformed space, let the number of cells in the radial direction also be $N$, but let the cell size in this space increase exponentially from the inner to the outer boundary until it become equal to $\Delta\rho$ (same as that in free space) at the outer boundary. (The advantage of this choice will be pointed out later).

If we define $\tau$ to be the multiplicative factor of a geometric series, we can express the cell sizes in the transformed domain as:

$$\Delta\rho'_i = \Delta\rho \cdot \tau^{i-1}$$

(25)

where $i = 1 \sim N$, and $i = 1$ corresponds to the outermost layer. We can determine the value of $\tau$ by using the relationship:

$$\sum_{i=1}^{N} \Delta\rho'_i = \Delta\rho \sum_{i=1}^{N} \tau^{i-1} = b - a$$

(26)

which leads to the equation:

$$\frac{1-\tau^N}{1-\tau} = \frac{b-a}{\Delta\rho}$$

(27)

Next, we take a cue from (20) to determine the relative permittivity $\epsilon_r$ by choosing:

$$\epsilon_{r,i} = \tau^{-(i-1)}$$

(28)
Let us consider a case example where the parameters of the geometry in Fig.5 are: $c = 1; \, b = 10; \, a = 5$ and $\tau = 0.8597$. Fig.5(c) shows the relative permittivity of the cloak derived by using (28). We can now see that the condition we imposed earlier on the cell size at the outermost layer of the transformed space, namely that it be equal to $\Delta \rho$ there, leads to $\varepsilon_r|_{\rho' = b} = 1$. This in turn guarantees that the cloak region will transition into the free space without an impedance mismatch, which is obviously desirable. If we let $c \to 0$, we can see that $\varepsilon_r|_{N} = \tau^{-(N-1)}$ at the inner boundary will be large; in fact, we can show that it will tend to very high values as $N \to \infty$.

At this point we return to Fig.1(c), where the results for the material parameters have been quoted from [5] for the same 2D problem, with identical polarization and geometrical parameters. It is interesting to note, first of all, that these medium parameters—both $\varepsilon$ and $\mu$—are dramatically different from those given in Fig.1(b) for the same identical geometry, although those in Fig.1(b) are presumably valid for an arbitrary angle of incidence and polarization of the incident wave. This is a somewhat surprising, since one would expect from the laws of Physics that the medium parameters would go smoothly from the set in Fig.1(b) to those in Fig.1(c) as the incidence angle changes from oblique to normal, and not in a discontinuous manner as shown in these figures. The second observation we make is also quite interesting, namely that, as mentioned previously, the result for $\varepsilon_z$ is identical to what we have found by using the WEB approach. Furthermore, $\mu_\phi$ is also equal $\mu_0$ in the TO case, which is again identical to our choice of $\mu$, which we have set to be isotropic at the outset. We point out that the material parameters in Fig.1(c) were obtained by applying the TO first, and then following it up with some simplifications after the fact (see [5] for details).

Of course, it is evident that, despite the similarities, the WEB results are not the same as those from the TO, since the approximated TO results still call for an anisotropic $\mu$, for which $\mu_r$ not only goes to 0 at $r = a$, but remains less than 1 throughout the cloak region except at the outer boundary, where it approaches $\mu_0$. It goes without saying that such an anisotropic $\mu$ would be extremely difficult to realize in practice; furthermore, the cloak designed with the above parameters would fail to work when either the incidence angle is oblique or the polarization is arbitrary, even if we were successful in realizing $\mu$ values that are less than 1, and those that tend to 0. Hence such a cloak would be of limited practical use, not only because of the above shortcomings mentioned above, but also because it is likely to be very narrowband when realized by using Metamaterials. The fact that these observations are indeed true has been verified experimentally, and we mention this in support of the statement that we have given above.

We next ask the question: “Is the WEB solution the complete answer to our cloak design problem?” After all, we have shown that the $E_z$ satisfies the same wave equations before and after the transformation from the $\rho$ to $\rho'$ systems. A short answer is NO, because even though we have satisfied the wave equations in the two systems, we have not imposed the same boundary conditions at the outer boundary in the transformed system as those in free space, where the cloak transitions into free space with a discontinuity in the material parameters, which is bound to produce reflections from the interface, unless of course the outer boundary of the cloak recedes to infinity, which is unrealistic. As we well know, the solution to a differential equation, such as the wave equation, is not unique until the boundary conditions are stipulated, which explains why the WEB solution we have derived in the cloak domain is not the same as that in free space; after all, the two satisfy different boundary conditions at $\rho = b$.

Though not shown here because of lack of space, we have also experimented with non-linear functions for mapping $\rho$ into $\rho'$, in which the $\varepsilon$ in the $\rho'$ systems transitions from a relatively high value at the inner boundary, similar to the $\varepsilon$ profile shown in Fig.1(b), for the general TO cloak. For instance we have chosen $\varepsilon_r = 275$ at the innermost boundary, which transitions exponentially to $\varepsilon_r = \varepsilon_0$ at the outer boundary to help mitigate the impedance matching problem at the interface with the free space, while we have still assumed that $\mu = \mu_0$ as we did for the case of
linear transformation to circumvent the realization problem. Unfortunately, we find that this WEB design for the cloak also fails to deliver a satisfactory performance, leading us to conclude that neither the TO nor the WEB approach based on scaling provide workable “practical” solutions to the cloaking problem, for the reasons we have already given above that have to do with the realization problems, as well as with matching issues if we introduce approximations of the material parameters or sudden truncations that produce reflections at the outer boundary. With this background in mind, we will introduce a different performance metric for the cloak in Sec.3 in order to address this issue.

Before closing this section, we mention that, for the general case where the mapping relationship takes the form:
\[ \rho' = f(\rho); \quad \phi' = \phi; \quad z' = z \] (29)
we define
\[ \rho = f^{-1}(\rho') \] (30)
where \( f^{-1} \) is the inverse function of \( f \), and the Helmholtz equation of the E-field can be rewritten as
\[ \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + k_0^2 E_z(\rho, \phi) = 0 \] (31)

The above equation can be written in the form
\[ \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + k_0^2 E_z(\rho, \phi) = 0 \] (32)

and there is no obvious way to choose the \( \varepsilon \) in the cloak domain—as we were able to do in the case of linear transformation from \( \rho \) to \( \rho' \)—so that the cloaked target would perform as though it was invisible, provided the matching issue is resolved, of course. We have already mentioned that assuming an \( \varepsilon_r \) variation of the form which transitions from 275 at \( \rho = a \), to 1 at \( \rho = b \), also fails to provide the desired solution.

Once the \( E \)-fields in the two domains have been matched, the corresponding \( H \)-fields which can be obtained from:
\[ H_\rho(\rho', \phi) = -\frac{1}{i\omega\mu_0} \frac{\partial E_z(\rho', \phi)}{\partial \phi} \] (33)
\[ H_\phi(\rho', \phi) = \frac{1}{i\omega\mu_0} \frac{\partial E_z(\rho', \phi)}{\partial \rho} \] (34)
will follow suit, and will match as well at the outer boundary if the cloak parameters tend to those of free space at the outer boundary, where it transitions into the external region without a mismatch. In fact, it is sufficient to match the \( E \)-fields only at the outer boundary, since we can invoke Huygen’s principle to argue that both \( E \)- and \( H \)-fields will match in the external region if we have done this.

In the general case of the transformation from one co-ordinate system \( (\xi_1, \xi_2, \xi_3) \) to another, say \( (\xi_1', \xi_2', \xi_3') \), the Laplacian takes the general form
\[ \nabla'^2 \mathbf{E} = \sum_k \sum_j \frac{1}{\partial \xi_k / \partial \xi_j} \left[ J B_{kj} \frac{\partial \mathbf{E}}{\partial \xi_j} \right] \] (35)
where
\[ J = \text{det} \begin{bmatrix} \frac{\partial \xi_1}{\partial \xi_1} & \frac{\partial \xi_1}{\partial \xi_2} & \frac{\partial \xi_1}{\partial \xi_3} \\ \frac{\partial \xi_2}{\partial \xi_1} & \frac{\partial \xi_2}{\partial \xi_2} & \frac{\partial \xi_2}{\partial \xi_3} \\ \frac{\partial \xi_3}{\partial \xi_1} & \frac{\partial \xi_3}{\partial \xi_2} & \frac{\partial \xi_3}{\partial \xi_3} \end{bmatrix} \] (36)
and
\[ B_{kj} = \frac{\partial \xi_j / \partial \xi_k}{\partial \xi_1 / \partial \xi_1} + \frac{\partial \xi_j / \partial \xi_k}{\partial \xi_2 / \partial \xi_2} + \frac{\partial \xi_j / \partial \xi_k}{\partial \xi_3 / \partial \xi_3} \] (37)
for orthogonal systems, when \( k \neq j \),
\[ B_{kj} = 0 \] (38)
and the Helmholtz equation is written as,
\[ \nabla'^2 \mathbf{E} + k^2 \mathbf{E} = \frac{1}{\partial \xi_3 / \partial \xi_3} \left( J B_{11} \frac{\partial \mathbf{E}}{\partial \xi_3} \right) + \frac{\partial}{\partial \xi_2} \left( J B_{22} \frac{\partial \mathbf{E}}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_1} \left( J B_{33} \frac{\partial \mathbf{E}}{\partial \xi_1} \right) + k^2 \mathbf{E} = 0 \] (39)
While there are no obvious ways we can use (39) in this case to determine the effective $\epsilon$ values in the transformed (cloak) domain, we can make locally 2D approximations and apply the scaling relationship which we have derived above. Finally, we mention that if we are successful in rendering the two $E$-fields equal in the two domains at the outer boundary, then the corresponding $H$-fields in the external regions will follow suit, as we can show from Maxwell’s equations; hence, we argue that it is sufficient to work with either the $E$- or $H$-fields, whichever is convenient for the problem at hand.

In summary, the WEB approach differs from the traditional TO approach in the following ways: (a) the WEB approach works with the Helmholtz form of the wave equation for either the Electric or the Magnetic field, rather than with Maxwell’s equations for both. Consequently, when we investigate incident waves of one polarization—exactly the same polarization investigated in a practical realization of the TO approach—the permittivity and permeability of the medium are isolated, and are required to have similar variations, typically anisotropic. In contrast to the TO equations, the permittivity and permeability appear as a product in the $k^2$ term in the WEB approach, which enables us to choose non-magnetic materials and facilitates the practical realization of the cloak significantly; (b) the impedance matching property of the TO-dictated material parameters is rendered ineffective when the condition of practical realization is enforced and a reduced (approximate) form of the anisotropic material parameters is employed. However, in the WEB approach the boundary condition can be defined, without making any approximations, in a way such that the permittivity of the cloak medium is matched at the cloak-medium-air interface, in order to ensure that there is no reflection at the boundary.

III. Conformal Mapping Approach

In this section we will discuss an alternate form of WEB approach, which is based on the Schwartz-Christoffel type of conformal mapping, and which provides a systematic way, not only to transform the wave equation, but also to impose the condition that the permittivity approaches $\varepsilon_0$ at the outer boundary.

It should be pointed out that if we choose $\mu_0 = 1$, as we have done in the present case, the magnetic fields will not satisfy the mapping relationship, even if the electric fields do. Hence we would need to ensure that the $H$-field in the transformed domain will tend to the $H$-field in the original space as the observation point moves to the outer boundary of the truncated cloak. To satisfy this requirement, we should not only focus on the transformation of the wave equation, but should also pay a close attention to the boundary condition. Below we show how we can use the Schwartz-Christoffel (SC) type of conformal mapping to ensure the impedance matching as well as the continuity of the fields at the boundary by appropriately choosing some of the transform parameters in the SC mapping.

Although the conformal mapping approach to transforming the wave equation is strictly limited to two-dimensional (2D) problems, we can again gain much insight from this example and draw some conclusions that are also applicable to the three-dimensional case. As we pointed out earlier, similar two-dimensional cases, e.g., bumps in PEC planes have been researched earlier [29] by using the TO algorithm, and carpet cloaks are but one of these examples.

We consider a triangular bump in a PEC plane, as shown in Fig. 6(b), which is defined by the coordinates $(u, v)$, so that we can transform it conveniently to or form a flat plane in the $(x, y)$ system, as in Fig. 6(a), by using the well-known Schwartz-Christoffel transformation [30, 31].

We assume, in common with the existing literature, that the fields are independent of the longitudinal direction $z$, and that all the fields can be derived from the $E_z$-component of the Electric field, which is the only non-zero component of the Electric field in (4).

In common with Sec. 2 we consider the TM$_z$ case once again, and set $E = 2E_z$. Next, as before, we eliminate $H$ from the curl equations in (4) to arrive at (9), where we have assumed that $\mu$ is simply $\mu_0$ in anticipation of the fact that we would search for dielectric-only materials when we carry
out the coordinate transformation of the Helmholtz equation (wave equation).

Our next step is to conformally map the \((x, y)\) domain into the \((u, v)\) system by using the Schwartz-Christoffel (SC) transformation, such that the flat plane in the \(\zeta = (x + jy)\) coordinate system (free-space domain) maps into the triangular bump in the \(\omega = (u + jv)\) coordinate system (physical space and cloak domain), and vice versa. The SC transform relationship reads:

\[
\omega = C \int_{0}^{\zeta} (\xi + k)^{-1/3}(\xi)^{2/3}(\xi - k)^{-1/3} d\xi + C_1
\]

where the parameters \(C, k, C_1\), are yet to be determined. A systematic approach for doing this will now be presented below.

From Fig.6 we see that the three points \(\zeta_1 = -k, \zeta_2 = 0, \zeta_3 = k\) in free space are transformed to the corresponding points \(\omega_1 = -1, \omega_2 = \sqrt{3}j, \omega_3 = 1\) in the physical space. Letting \(\zeta_2 = 0\) and \(\omega_2 = \sqrt{3}j\) in (40), we can readily obtain \(C_1 = \sqrt{3}j\). In view of the symmetry, either of the points 1 and 3 can be chosen to determine the values of \(k\) and \(C\). Using \(\zeta_3 = k\) and letting \(\omega_2 = 1\) in (40), we get:

\[
|C| e^{i\theta} \int_{0}^{k} \frac{\xi^{2/3}}{(\xi^2 - k^2)^{1/3}} d\xi = 2e^{-j\frac{\pi}{3}} \quad (41)
\]

By taking the modulus of each side of (41), we obtain:

\[
|C| \int_{0}^{k} \frac{\xi^{2/3}}{(\xi^2 - k^2)^{1/3}} d\xi | = 2 \quad (42)
\]

The choice of \(k\) in (42) appears to be arbitrary at this point if we do not impose any condition on \(|C|\). And, in fact, if we use the SC-Tool of MATLAB code [32], it makes the default choice \(k = 1\), which leads to \(|C| = 2.3192\).

However, we will soon see that this choice of \(k\) is not suitable for our purpose, since the asymptotic behavior of \(d\omega/d\zeta\) is not what we desire for it to be. To explain why this is the case we insert a box-shaped boundary in the \((u, v)\)-domain as shown by dotted line in Fig.6. Since, from a practical point of view we must truncate the cloak with which we are going to cover the triangular bump, we define the outer boundary of the cloak so that it coincides with the surface of the box. We will show below that by introducing appropriate dielectric-only materials in the cloak region of the \((u, v)\)-plane, we can render the \(E_z\)-field in the transformed (cloak) domain to be the same as that in the half-space of the \((x, y)\)-plane, which is free space. To show this, and to determine the material parameters of the cloak region, we turn to the Helmholtz equations of (9) in the free space of \(\zeta\) domain rewrite as:

\[
\nabla^2_{\zeta} \Psi(x, y) + k_0^2 \Psi(x, y) = 0 \quad (43)
\]

In the \(\omega\)-domain, the medium parameter are \(\mu_0\) and \(\varepsilon_r\varepsilon_0\), where \(\varepsilon_r\) is yet to be determined. Then \(\Psi(u, v)\) satisfies:

\[
\nabla^2_{uv} \Psi(u, v) + k_1^2 \Psi(u, v) = 0 \quad (44)
\]

where

\[
k_1^2 = k_0^2 n^2 = (2\pi f)^2 \mu_0 \varepsilon_0 \varepsilon_r = k_0^2 \varepsilon_r \quad (45)
\]

We can transform (44) into:

\[
\nabla^2_{x,y} \Psi(x, y) + k_0^2 \varepsilon_r (d\omega/d\zeta)^2 \Psi(x, y) = 0 \quad (46)
\]

To render (46) equal to (43), we let

\[
\varepsilon_r = (d\omega/d\zeta)^{-2} \quad (47)
\]

This is the choice for the material parameters we need to make in the \(\omega\) (cloak-domain), so that the field solution in this domain is identical to the one in the \(\zeta\) -domain, i.e., \(\Psi(x, y)\), which comprises only the specularly “reflected” field from the PEC plane, and which implies that the scattered field is zero. We can then argue that with the choice of (47) for \(\varepsilon_r\), the cloaked bump on a
Since \( |d\omega/d\zeta|_{\omega\to\infty} = |C| \), we must choose \( |C| = 1 \) if the boundary conditions at infinity are to be the same. This implies, in turn, that \( \epsilon_r|_{\omega\to\infty} = 1 \). Furthermore, if the region external to the “carpet” is to be free space so that the cloak can be truncated, the permittivity of the carpet should approach the permittivity of free space to minimize any reflection at the interface. This also requires that we enforce the condition \( |C| = 1 \).

Now we can understand why we mentioned earlier that the choice of \( |C| = 2.3192 \) is not good for designing a cloak with minimum reflection at the interface without the need of a matching layer, which we required for the case example presented earlier in Sec.2 for cloaking the cylindrical object designed by using the WEB approach with a linear transformation relationship between \( \rho \) and \( \rho' \). Fortunately, for the case of conformal mapping we can adjust the value of \( |C| \) by choosing an appropriate \( k \). For instance, using (40), we can show that if \( k = 2.3192 \), \( |C| = 1 \), and the truncation condition \( \epsilon_r|_{\omega\to\infty} = 1 \) is indeed satisfied.

Figs.7(a) and (b) display the meshes in original and the transform spaces. Fig.8 presents the material (\( \epsilon \) variation) that we need to realize for the carpet. We note that \( \epsilon_r \to \infty \) at the tip of the triangular bump, and is equal to zero at the two points \( u = \pm 1, v = 0 \), where the triangular bump joins up with the PEC plane. It is obvious that these parameter values would be difficult to realize, even though unlike TO the required \( \epsilon_r \) values are isotropic in this case. Even if we were to use a bump with a smooth curved profile to replace the triangular bump, it will still require \( \epsilon_r < 1 \), which is difficult to realize.

Obviously, strictly speaking, such a cloak is not physically realizable, not only at the tip region, but where the bump joins up with the PEC plane and the required \( \epsilon \to 0 \).

In any case, we have seen that even for the example of the simple cylindrical geometry presented above, the material parameters required for the cloak designs based on TO are difficult to realize, whether they are exact (see Fig. 1(b)) or approximate (Fig.1(c)). Additionally, we note that at \( \rho' = b \), which is the outer boundary of the cloak, there can be a discontinuity in the material parameters, if we make approximations such as assume \( \mu = \mu_0 \), as we did in the WEB design, or if we choose simplified parameters for \( \epsilon \) in TO or WEB designs for the sake of realization. This is because they might produce untoward reflections of the incident wave impinging upon the cloaked cylinder from outside of the cloaked region.

In summary, there are two major problems with both the TO and WEB approaches to cloak design, where we seek to reduce the radar signature of the target in all directions (bistatic), and to make it invisible in the limiting (ideal) case. First of these is the material realization problem, despite the fact that we only need isotropic and homogeneous dielectric materials in the present WEB-design, as opposed to inhomogeneous and highly anisotropic materials called for in the TO-
design, as we have pointed out before. The second problem is associated with the boundary; whose choice appears to be arbitrary. It is claimed in TO, that this problem is circumvented by requiring $\tilde{\epsilon} = \tilde{\mu}$; however, that solution is hardly realistic from a practical point of view, since realizing even isotropic $\mu$ values with the required range of parameters is a formidable if not impossible task for TO-based designs, and the situation with the realization of $\epsilon$ is not much better either. This is evidenced by the fact that various approximations have been suggested [5] to get around this difficult issue—after the initial design has been found to be fraught with realization problems—albeit at the expense of compromising the performance of the cloak. In principle, we could address the matching issue in the WEB design, but the realization issue would still persist. For this reason, in Sec. 3 we will propose a different performance metric for the cloak, which is widely used by the microwave community, when we discuss the Scattering matrix approach to formulating the problem at hand.

We will close this section by providing one more example which underscores the importance of satisfying the boundary condition at the interface between the transformed and external regions. Towards this end, let us consider the example shown in Fig. 9 to illustrate our point. On the left side of this figure we see a sectoral horn whose aperture size is $a$. Let us say we map the geometry of this horn to the one on the right whose aperture dimension is shorter, say $b$, where $b < a$. Then, if we were to conjecture that we can just transform the inner regions (shaded), and find materials that make the fields in the two shaded domains the same, say by using either the TO or WEB algorithms, we would think that we could get the horn with the small aperture (right) to perform just the same as the one with a larger aperture (left) and to provide the same gain values. While it would be very desirable to reduce the aperture size of an antenna and achieve super gain characteristics without compromising its performance by using the TO algorithm, unfortunately this turns out not to be possible. This is because to ensure the field equivalence between the two domains we must transform the entire region in which the horn is embedded, and not just the shaded region. The material parameters of the horn on the right would extend well into its external region and will not be confined to the interior (shaded) region alone, in order to correctly map the fields from the left to the right figures, or vice versa. Consequently, the true aperture size of the horn in the right will in general be larger than the opening of its mouth, and will depend on where we place the truncation boundary of the loading materials. Note that, once again, the boundary condition will play an important role, and we will face the problem of reflection from the interface if the medium parameters do not tend to the free space values, namely $\epsilon_0$ an $\mu_0$, at the outer boundary, or the impedance is matched some other way, e.g., by using anisotropic materials with parameters that are difficult to realize in practice.

**IV. Generalized Scattering Matrix Approach**

In the previous sections we have discussed two different WEB (wave-equation-based) approaches for designing cloaks and carpets that can reduce the scattering from certain types of objects, or even render them invisible, by using dielectric materials alone. We have found that material (relative permittivity) parameters required to accomplish this can vary over a wide range; furthermore, in the TO designs, $\epsilon_r$ and $\mu_r$ can either be much smaller than 1, much greater than 1, or combinations thereof, depending on the geometry of the object. We point out that this problem is compounded in the TO-based approaches because there both $\epsilon$ and $\mu$ are tensors in general, and this makes it even harder to realize them, even using Metamaterials. It is not uncommon therefore, to find designers
introducing approximations and replacing the TO-dictated materials with much simpler ones (see [5] for instance) that deviate substantially from the TO-specified ones in (48), so that they are more realistic in terms of realization potential than are the original TO ones expressed in (49), albeit at the expense of compromising the performance in comparison to that achievable by using the ideal set of parameters. In [5] these modified parameters are:

\[
\begin{align*}
\varepsilon_{\rho} &= \mu_{\rho} = \frac{\rho - R_1}{\rho} \\
\varepsilon_{\phi} &= \mu_{\phi} = \frac{\rho}{\rho - R_1} \\
\varepsilon_{z} &= \mu_{z} = \left( \frac{R_2}{R_2 - R_1} \right)^2 \frac{\rho - R_1}{\rho} \\
\mu_{\rho} &= \left( \frac{\rho - R_1}{\rho} \right)^2 \\
\mu_{\phi} &= 1 \\
\varepsilon_{z} &= \left( \frac{R_2}{R_2 - R_1} \right)^2
\end{align*}
\] (48)

which are significantly different from the original ones dictated by the TO, and are plotted in Fig.1(b).

The root cause of this difficulty can be linked to the fact that in the TO-paradigm we insist that the field quantities in the original and transformed systems be identical, as we do when we are locked into the TO-based paradigm. In this Section we propose a different performance metric in the context of Generalized Scattering Matrix (GSM), which we introduce below. We show that the TO prescription of field equivalence between the original and transformed systems is a special case and that the GSM provides us more flexibility than does the TO, by modifying the performance metrics of the electromagnetic device that we are designing, dramatically increasing its chances for practical realization by using available materials. Thus although the performance of the device following the GSM prescription will be less than ideal, it may be quite satisfactory from the point of view of practical applications that we may have in mind. We will provide several examples to illustrate this by considering RCS-

Fig.10. Generalized Scattering Matrix approach in the context of the Field Transformation (FT) method reducing blankets, flat lenses, reflect arrays, etc., all of which face the realization problems when using the TO approach, which calls for anisotropic materials (both \(\varepsilon\) and \(\mu\)), and whose realization has been elusive at best.

To introduce the GSM approach [26, 33] in the context of the so-called Field Transformation method [34], we refer the reader to Fig.10, where we have defined the input and output ports to correspond to interfaces that bound an electromagnetic device. The field distribution in the input port, which is illuminated by the source located at the left of the port, can be expressed in terms of a set of coefficients \(a_i^1\) (vector) associated with the basis functions used to represent this “incident” field in the absence of the device when there are no reflections. A convenient choice for the set of basis functions is the discrete plane wave spectrum when we are dealing with objects in free space, where the discrete plane waves play a role similar to those of the set of waveguide modes in “closed” guided wave structures.

Next, we insert the electromagnetic device, whose Scattering Matrix we desire to describe, inside the region bracketed by the input and output ports. We define a set of coefficients \(b_i^1\), again associated with the same basis functions as we used to define \(a_i^1\), to represent the outgoing fields scattered by the electromagnetic device, i.e., the “reflected” fields that originate from the device and propagate back towards the source. We can similarly define a set of coefficients \(c_i^1\), associated with the field distribution in the output port, through which these fields propagate in the free-space region to the right of this port, and are termed the “transmitted” fields. Our next
step is to place the illuminating source to the right of the output port, which we have previously defined when the source was at the left, and reverse the roles of the input and output ports to correspond to the new source location. The incident, reflected, and transmitted fields are now characterized by a new set of coefficients $a^2_i, b^2_i,$ and $c^2_i$, where $c^2_i$ fields now propagate to the left of the device, whereas the $b^2_i$ fields do the opposite, i.e., propagate to the right.

Below we present several examples of practical devices where we use the GSM to define their performance metric.

We consider a radar target whose RCS we wish to reduce by covering it with a suitable material, which we can either fabricate in the Lab or which is commercially available, so we do not have to wrestle with the feasibility issue in the design process later. The performance metric we define is that the “magnitudes” of the elements of $S_{11}$, comprising of the scattered fields from the target in the left hemisphere be below a certain threshold level. It is important to note that we do not impose any condition on the fields scattered in the forward direction, i.e., on $S_{12}$, either its magnitude or its phase, neither do we stipulate any condition on the phase behavior of the elements of $S_{11}$. This is in contrast to TO, which aims for the ideal situation where the magnitudes and the phases of both $S_{11}$ and $S_{12}$ at the input and output ports, respectively, be identical to the same as they would be if the target were invisible, or was absent altogether, implying that the fields scattered by the target to the left of the input port or to the output port be all identically zero.

We are now ready to define the scattering matrix $[S]$, via (50) below, which we will use to characterize the device, as follows:

$$b = [S]a \quad (50)$$

or explicitly,

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad (51)$$

where $b = \{b^1, b^2\}$ and $a = \{a^1, a^2\}$ represent the weights of the outgoing and incoming field representations at the input and output ports, respectively. Equation (51) provides us a convenient way to characterize an electromagnetic device in terms of its response to a plane wave, regardless of whether the illuminating source is incident from the left or the right of the device.

It is not difficult to see that the GSM metric is considerably easier to realize than the corresponding TO-dictated metric we just mentioned. Thus if in the radar scenario it is deemed sufficient to be concerned only with the backscattering region, the GSM metric will suffice, and we can use available [35-37] electric and magnetic materials to design a thin coating for the target (typically only a few millimeters thick) which will reduce the scattering (RCS) over a very wide frequency band, and for arbitrary angle of incidence and polarization of the radar signal impinging on the target, which is highly desirable in practical radar applications. The multilayer coating essentially serves as a matching region which transforms the surface impedance of the target (zero for PEC scatterers) to the wave impedance of free space at the interface between the coating and external region, which is free space by choice because we wish to truncate the cloak, which is theoretically infinite. The transition between the coating and the external region is thus handled naturally in this scheme in one-step process.

We should mention here that attempts have also been made to make the coating thin for TO-based designs, albeit for simple cylindrical-shaped targets of infinite length, for the case of radar signal at normal incidence ($\theta = 90^\circ$). We present the results of numerical simulations below that show that the performance of three-layer discretization of the thin cloak is not very satisfactory.

We examine the problem of reducing the level of scattering from radar targets—a problem that has been extensively researched into in the context of the TO. Once again, we will cast the problem in the language of Scattering Matrices, to help us understand why the TO approach leads us to untenable situations and/or to solutions which call for MTMs that have many drawbacks as we have pointed out earlier. We will also show how the FT approach mitigates the problems alluded to above by restating the design objectives and
Let us consider an arbitrarily shaped radar target placed in the region between the input and output ports, as shown in Fig. 11. Next, let us suppose that our objective is to reduce the level of scattering from the target, both in the forward and backward directions to the extent that the target becomes totally invisible to the incident field, say from an interrogating radar. We could cast this objective in the language of the Scattering matrices, by specifying that \( S_{11} \) be identically zero at the input port and \( S_{12} \) be such that the field distribution at the output port is identical to the incident field, as though the scatterer was totally transparent or invisible.

To circumvent the problems that we encounter when attempting to use the TO in order to meet the ideal but unrealistic goal of making the target altogether invisible, we turn to the FT (Field Transformation) approach and modify our stated objectives, again in the context of S-parameters. In contrast to the TO, this time we ask that \( S_{11} \) be small—in terms of magnitude only—but not 0, as we demanded in the case of TO. Furthermore, we do not impose any restrictions on \( S_{12} \), as we did in the case of the TO when we stipulated that the scattered fields at the output port be identically zero, so that the total field there be just the incident field in the absence of the target. While we concede that the performance of the FT-based cloak or the blanket won’t be as ideal as the performance of the TO-based design would have been if we could realize it, we certainly stand to gain considerably when we follow this strategy, since we can now obtain realizable solutions for the cloak that are very wideband and can cover the entire frequency range of 2-18 GHz, for instance, if we so desire. Furthermore, the cloak (or blanket) can now be very thin (only a few millimeters) and it would work for arbitrary incident angles and polarizations as well. In short, we can say that this strategy of following the FT-based algorithm to design the cloak enables us to circumvent all the problems we encounter when employing the TO-based design strategy instead. Of course, we compromise the performance of the cloak in the forward scattering direction when we use the FT-based strategy, though that is not a problem for either monostatic or bistatic radars that are only concerned with the backscattered fields.

Some examples of such realistic blanket designs based on the FT strategy can be found in [38] and are omitted here for the sake of brevity. The above work also shows how to design the blankets for arbitrary targets by borrowing some ideas from the TO algorithm to improve their design.

Before closing this discussion, we would like to mention that the strategy for designing carpet cloaks is very similar. We can use the image theory to recast the problem of designing a carpet.
cloak so that it totally equivalent to the cloak design problem for radar targets that we have discussed above in this section.

As is well known, the RAMs (radar absorbing materials) have been around for a very long time, some for many decades, dating back to when stealth aircrafts came into vogue in the sixties, and earliest theoretical and experimental work starting around 1930s. We realize that information on some of these RAM materials is not available openly because of their “classified” or “secret” nature, understandably so because they are used in military applications to design stealth aircrafts and missiles. Nonetheless a plethora of information about similar absorbing materials is available in the open literature, including the details of their fabrication, which have been described in [35, 36], for instance.

Here we will use two different types of materials namely CoFe Nano-Flakes (NF) and CoFe Nano-Particles (NP), whose frequency variations are shown in Fig.13. We point out that these materials can be realized with relative ease, as is evident from [35, 36], where the details of their fabrication can be found.

We note that the $\varepsilon$ and $\mu$ values of these materials are both complex and dispersive. We hasten to point out, however, that unlike the MTMs, which typically have variations with frequency because of resonant inclusions they use to achieve effective $\varepsilon$ and $\mu$ values that are very high, very low, or even negative, the material parameters of the NP and NF materials vary relatively smoothly with frequency, and this is crucial for realizing wideband performance, as we will soon see.

To illustrate the fact that we can indeed achieve wideband performance in terms of reflection reduction over a wide frequency band with relative small thicknesses of 2, 4, 6 and 7 layer absorbers we refer to Fig.14, in which a 10dB (or better) reduction in the reflection coefficient is presented. Although not shown here, the results for the reflection coefficient reduction are also satisfactory when either the polarization, or the incident angle is varied and this is also true when both are changed simultaneously.

We now move to the second step in our design procedure, which is to adapt the blanket designed for the infinite PEC plane to an arbitrarily shaped object. Initially we consider an object with a smooth surface whose radius of curvature is moderate-to-large everywhere. We will generalize the procedure in the third step, using the principles of the TO when the above assumption regarding the smoothness of the object is not valid, as for instance when the object has sharp edges or bumps, as a general target would in practice.

When the object has a relatively smooth geometry, we initially wrap the multilayer
absorbing blanket, which we have designed earlier for the planar surface around the PEC target whose scattering cross-section we are attempting to reduce, and test the effectiveness of the blanket for the new object. For a wide variety of targets that we have examined, a number of which are shown in Fig. 15, we have found that the blanket does reduce the monostatic as well as the bistatic radar cross-section in the “reflection” region near the surface of the object for different angles of incidence and polarizations of the incoming wave. The results for a two-layer absorber are shown in Figs. 16 and 17 for a rectangular cylinder of finite length, which we have studied as a test case.

A simple test, which is typically applied to cloak designs, is to examine the wavefront of the total (incident+scattered) field, and see how the level of distortion of the wavefront decreases when the scatterer is covered by the layered absorbing coating. We present the plots of these wavefronts of the total fields for normal and oblique incidence cases for both polarizations in Figs. 16 and 17, respectively.

We observe that the object, which is a finite cylinder of height 18cm, generates distorted wavefronts owing to the contribution of the scattered field from the object, even when covered by a two-layer blanket, designed for the infinite planar PEC object, for the nominal frequency range of 4.6-18 GHz, with a nominal reflection coefficient of -10dB or less. However, we also note from Fig. 16 that the distortion in the phase front is relatively small once we go above the low-end of the design frequency range, viz., 4.6 GHz for the planar geometry, confirming that the planar design performs reasonably well even though we are dealing with a rectangular cylinder now. We hasten to point out that the results presented in Fig. 16 are not for a coating which has been optimized for the object at hand, and we expect some compromise in the performance of the coating. However, we can improve this performance by optimizing the parameters of the
two-layer design, specifically the relative thicknesses of the two-layer, even as we maintain the total thickness intact. We expect the changes to be relatively minor, however, except for the corner regions and, hence, the optimization process should be realistic as well as numerically feasible. The above remarks are also applicable to the oblique incidence case, for which some sample results are presented in Fig.17.

It is important to point out that the strategy for designing the absorptive coating, presented herein, is very different from that employed for ideal traditional TO cloak, since the latter is designed to render the (object+cloak) composite to have a zero scattering cross-section in all directions, whereas the blanket design introduced here seeks to reduce both backscattering and bi-static scattering scenarios but only in the reflection region, and not in the forward-scattering direction. We hasten to point out, however, that this type of performance is perfectly well suited for modern radar systems, symbolically depicted in Fig.18 above, where only the scattering in the reflection region is of concern.

For the final step, we consider the problem of absorber design when a shape perturbation is introduced in an object. Let us say that our modified target is the same rectangular cylinder we just considered above, except for a bump on the top surface. The extra corners introduced by the perturbation, be they smooth or sharp, would obviously introduce additional distortions in the planar phase front, and potentially increase the scattering level. Our objective here is to restore the field behavior so that it is close to that of the original object that we had prior to the introduction of the perturbation.

We now outline the procedure for the blanket design for the new object, shown on the left in Fig.19, i.e., Fig.19(1), which is in the physical domain, and is a modified version of the one shown in the right side of the same figure; i.e., Fig.19(2), which corresponds to the virtual domain. Note that unlike the circular cylinder example we discussed earlier, the medium in the virtual domain, surrounding the object, is no longer free-space, as was the case shown in Fig.18. Note also that the dimensions of the objects in the two domains are comparable, and are totally different from the legacy TO-design case, in which the scale factor between the dimensions of the object in the physical and virtual domains tends to infinity to render the target invisible.

The field behaviors for the perturbed object with locally modified material parameters can be seen from Figs.20 and 21. Fig.20 shows that the amplitude of the scattered E-field is reduced, and that the phase front of the total E-field is approximately restored as well in the reflection region. Fig.21 compares the amplitudes of the scattered electric fields for five different scenarios listed in the figure.
Fig. 20. Phase behavior of scattered E-field for (a) perturbed object wrapped by blanket with original medium parameter; (b) perturbed object wrapped by blanket with locally modified medium parameter.

Fig. 21. Comparison of the amplitudes of electric fields scattered by different objects.

We should clarify the fact that although we are referring to this geometry as a slab, what we are really dealing with is a wide rectangular cylinder, with a small thickness.

We note that the modified slab does introduce additional scattering, and that the absorber does help reduce the same. We also note that the modified absorber improves the performance over the initial one, but only slightly, which shows that the planar version of the cloak is not all that inferior to the one modified for this type of geometry. Additional optimization of the modified cloak is expected to improve the performance even further, if so desired.

We look at an important application of RFI reduction, i.e., to mitigate the problem of antenna blockage in a shared-platform environment, as shown in Fig. 22, in which the introduction of the aggressor monopole antenna can raise the far-end sidelobe levels of the parabolic dish significantly. To mitigate this effect, we can wrap the monopole by using a multilayer absorber. Alternatively, we can use a discretized cloak structure to cover the monopole. Fig. 23 shows the far-field patterns of a parabolic dish antenna—the victim antenna—when the aggressor antenna, is placed in the vicinity of the dish. The patterns of the dish/monopole composite are also included in the figure for comparison with the two treatment plans applied to the aggressor antenna. We note that the introduction of the absorber treatment reduces the sidelobe levels of the dish antenna, as compared to the case for the far-end antenna combination without the treatment. We also note
that an elaborate cloak design, however, does not mitigate the pattern interference problem, or rather, aggravates it.

V. Metasurface and Lens designs

The subject of field manipulation is a topic of great interest today, as evidenced by a rather large number of publications on this topic in recent years. The present authors have discussed this topic elsewhere in a recent publication [33] in the context of the Generalized Scattering Matrix approach, which we have described above in Sec.4. Here we simply summarize the approach briefly and mention some representative examples for the sake of completeness.

We begin with the problem of designing a flat surface that mimics a parabolic reflector, which may be either symmetric or offset-fed. We assume that the reflector is illuminated in the usual manner, by using a feed horn which emanates a spherical wave from its phase center (see Fig.24). The reflector serves to convert this wave front to a planar one to generate a directive beam in a specified direction, which depends on the choice of the feed location.

The TO approach to handling the problem of designing the flat reflector is relatively straightforward: Transform the parabolic reflector geometry into a planar one and embed it in a medium that surrounds it so that the fields reflected off the surface of the planar reflector covered by the TO-designed coating mimic those associated with the original parabolic surface. The Jacobians of the transformation provide a relatively easy way to determine the parameters of the medium in which we must embed the flat reflector so that it delivers a performance that is identical to that of the parabolic reflector. But we encounter similar types of realization problems in the TO design of these surfaces, as we did when realizing cloaks, namely finding the desired low-loss, low-dispersive and wideband materials with which to cover the surface of the flat reflector. We note, however, that such surfaces that are also known as reflectarrays, are almost always designed to match only the phase of the reflected field, rather than to control the amplitude distribution of this field in its equivalent aperture as well. Thus, in the context of $S$-parameters, we can simply use a one-port $S$-parameter description (see Fig.6) since the surface is backed by a PEC plane, and require only the phase of $S_{11}$, i.e., that of the reflected field, to satisfy the desired specifications. We can then design the surface with isotropic and dielectric-only materials by locally controlling just the phase of the reflection coefficient. In this approach we can circumvent the need to use both $\epsilon$ and $\mu$, which would not only be anisotropic but would require the use of Metamaterials as well, if we follow the TO paradigm strictly and proceed to match the fields in the original and transform domains in their entirety.

Similarly, for the case of a flat lens which mimics a conventional convex lens, we impose the condition only on the phase of $S_{12}$, where we locate the input and output aperture planes just to the left and right of the aperture, respectively (see Fig.25). Here again, we can use isotropic dielectric-only materials and avoid the problems typically with the TO designs. We do not impose any condition on either the magnitude of $S_{11}$, or on the magnitude of $S_{12}$, though we would obviously wish to minimize $S_{11}$ for an efficient design.

For the last example we consider the problem of manipulating the amplitude distribution in the aperture of a feed horn to control its edge taper. While the primary use of the TO algorithm is to relate the material parameters of two transformed domains to each other such that the field solutions
in these two domains are identical, it is possible to generalize it to determine the material parameters of the intervening medium that would transform the field from the input to the output port (see Fig.10 in Sec.4) in a desired manner. As an example, we may wish to modify the aperture distribution of a horn antenna (see Fig.26) to achieve certain characteristics of its radiation pattern, e.g., the sidelobe level, which is determined by its edge taper of the distribution. We will now describe how we might control this edge taper by inserting a material slab in the aperture of the horn, or just above it. We will now discuss two different ways to accomplish this goal, namely to use the TO and Ray Optics approaches.

In the past, the TO has been used to control a beam in several ways, e.g., to bend and expand the beam [39], to shift and split [40], and to generate multiple beams [41]. In this section we will examine the question whether or not we can transform a given field distribution at the input port to the desired one at the output port by using the TO. As shown in [42], the field can be concentrated to have a higher energy density in a region by using appropriate material parameters dictated by the TO. In contrast to [42], we attempt to use a method that manipulates the field distribution profile by letting the fields propagate through a region in which a uniform mesh is modified to a denser or coarser mesh--depending upon whether we want a higher or lower amplitude in the output plane--and the cells are filled with media whose parameters are dictated by the TO. It should be pointed out that the mesh inside the slab (see Fig.27) should be carefully designed to ensure a smooth transition from the input to the output aperture.

As mentioned earlier, the field transformation via TO would typically call for the materials in the slab to be anisotropic. Even if we could realize the anisotropic lens designed by using the TO, we should recognize the fact that the performance of the lens would not be as we would desire for it to be. This is because we would have transformed only the fields inside the lens, and the fields outside the lens have not be...
transformed. As a result, there would be reflections at the boundaries of the lens because we have ignored the boundary conditions at the interfaces.

Before closing it is worthwhile mentioning, that the lenses and reflectarrays designed by using the present $S$-parameter-based approach are not only easier to realize in practice, but they have been found to exhibit superior performances as well when compared to TO-based designs.

It is worthwhile to point out it is possible to also achieve the desired field manipulation by utilizing isotropic dielectric materials only, which is highly desirable from realization point of view. The design of the slab, which acts somewhat like a flat concave lens in this case, has been carried out in [29] by using a ray-optical approach in the same manner as that employed for the flat lens design we have described above. Fig.28 shows an example of this type of design, where the rings with varying permittivity are used to construct a flat concave lens that covers the horn. The relative permittivity values of the rings, listed in Table 1, is seen to increase form center to the edge. This, in turn, introduces a larger phase delay at the edge than at the center, which results in an expansion of beam as shown in Fig.28(c), where the aperture distributions with and without the lens are presented. Note that, in contrast to the TO-based design, the medium in the present design is dielectric-only, is isotropic and requires only realizable materials rather than Metamaterials.

VI. Conclusions

In this work we have investigated two different Wave-equation-based (WEB) approaches, namely coordinate transformation and Conformal mapping, to mimic the Transformation-Optics-type (TO-type) field transformation between two coordinate systems that leave the electromagnetic fields intact. Our primary motivation was to see if we could mitigate the problems encountered during the realization of materials prescribed by the TO-based designs, which are anisotropic and which vary over a wide parametric range. We have found that although the material realization problems are less formidable with the WEB approach because the materials we need are isotropic, the WEB approach still faces realization problems and/or matching issues. Finally, we have presented an alternative to the TO-type of field transformation approach, which is based on the General Scattering Matrix (GSM) concept, and have shown how it can help us resolve the material realization issue provided we modify the performance metric of the electromagnetic devices from that employed by the TO-type algorithms.

References


Table 1. Relative permittivity values of rings of the flat concave lens.

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[34] R. Mittra, Y. Zhou, Designing Cloaks and Absorbing Blankets for Scattering Reduction


Raj Mittra is a Professor in the Department of Electrical & Computer Science of the University of Central Florida in Orlando, FL., where he is the Director of the Electromagnetic Communication Laboratory. Prior to joining the University of Central Florida, he worked at Penn State as a Professor in the Electrical and Computer Engineering from 1996 through June, 2015. He also worked as a Professor in the Electrical and Computer Engineering at the University of Illinois in Urbana Champaign from 1957 through 1996, when he moved to the Penn State University. Currently, he also holds the position of Hi-Ci Professor at King Abdulaziz University in Saudi Arabia.

He is a Life Fellow of the IEEE, a Past-President of AP-S, and he has served as the Editor of the Transactions of the Antennas and Propagation Society. He won the Guggenheim Fellowship Award in 1965, the IEEE Centennial Medal in 1984, and the IEEE Millennium medal in 2000. Other honors include the IEEE/AP-S Distinguished Achievement Award in 2002, the Chen-To Tai Education Award in 2004 and the IEEE Electromagnetics Award in 2006, and the IEEE James H. Mulligan Award in 2011.

Dr. Mittra is a Principal Scientist and President of RM Associates, a consulting company founded in 1980, which provides services to industrial and governmental organizations, both in the U.S. and abroad.

ZHANG Pengfei was born in 1979 in Shanxi Province. He received B.S. degree, M.S. degree, and Ph.D. degree in electromagnetic field and microwave technology from Xidian University, in 2001, 2006, and 2008, respectively. In 2007, he joined the National Laboratory of Science and
Technology on Antennas and Microwaves, and worked as an associate professor in this lab from 2009. This work was carried out when Penfei Zhang was at the University of Central Florida as a Chinese Scholarship Council (CSC) Scholar on leave from National Key Laboratory of antenna and Microwave Technology. His research and development interests include the design of lens antenna, prediction and control of Radar cross section (RCS), the design of water antennas, electromagnetic analysis and electromagnetic calculation. He is the author of over 20 referred journal and conference papers on these topics.