Fundamentals of Graphene Magnetoplasmons: Principles, Structures and Devices

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Abstract— This paper presents an overview of graphene magnetoplasmonics, including a description of related nonreciprocity and applications to devices such as isolators and couplers.

Index Terms — Graphene, surface plasmons, magnetoplasmons, nonreciprocity, nonreciprocal devices

I. INTRODUCTION

Graphene, a monoatomic layer material made of carbon atoms arranged in a honeycomb lattice, is a true 2D crystal structure. It had been known for decades that graphite, the material used in pencil tips, is made of stacked layers of graphene held together by weak Van der Waals forces. Moreover, the electronic properties of graphene and its relativistic band structure have been the object of intensive theoretical study in connection with the graphite used in nuclear reactors, [1]-[4]. However, before its first experimental realization in 2004 by a group at the University of Manchester in 2004 [2], it was believed that such a 2D crystal would be unstable due to thermal and other crystal fluctuations and could, therefore, not be practically obtained [3]. Following its aforementioned experimental validation and due to its unique electronic properties, graphene has triggered huge research efforts on its various features including electronic, electromagnetic, mechanical and thermal properties.

More recently, graphene plasmonics has become a research area of particular interest in electromagnetics, due to several unique properties [5]. Our group has developed applications involving nonreciprocity, which can be tailored towards nonreciprocal plasmonic devices such as plasmonic isolators [6]-[15]. One of the methods for generating nonreciprocity in graphene is through a static magnetic field. The resulting magnetoplasmons and their nonreciprocity are the focus of this paper, which aims at presenting the relevant material in a self-consistent and pedagogical fashion. A brief introduction on the fundamentals of graphene, its band structure and its conductivity model is presented in Sections II and III. Section IV includes introductory material on surface plasmons and magnetoplasmons in graphene, and finally, Section V focuses on nonreciprocal properties of graphene magnetoplasmons and their applications.

II. FUNDAMENTALS OF GRAPHENE

First, a brief overview of graphene carbon bonding and crystallography is presented. These concepts are then applied in the tight binding model to derive the energy band structure of graphene. Finally, the concept of graphene’s effective Dirac Hamiltonian is introduced, as a tool to determine the quantized energy levels in magnetically biased graphene.

A. Carbon bonds in graphene

The carbon atom, the sole constituent of graphene, has four electrons in its outer shell, called the valence electrons, occupying the 2s and 2p atomic orbitals. In the crystalline phase, these weakly-bound electrons give rise to 2s, 2px, 2py and 2pz orbitals, which are important for forming covalent bonds in carbon materials [16]. Since the energy difference between 2s and 2p orbitals in carbon is small compared to the energy of the chemical bonds, the electronic wave function for these four electrons can readily mix with each other so as to enhance the binding energy of the C atom with its neighboring atoms [16]. This mixing of 2s and 2p atomic orbitals is called hybridization, whereas the mixing of a single 2s electron with n = 1, 2, 3 2p orbitals is called spⁿ hybridization.

In the graphene 2D crystal, the outer shell electrons of the carbon atom form sp² hybridizations. The 2s and 2px and 2py orbitals mix to form three sp² orbitals in the xy plane (plane of graphene) with angles of 120 degrees, leaving behind an unaltered 2pz orbital normal to the xy plane, shown in Fig. 1.
In graphene’s honeycomb structure, these three \( sp^2 \) orbitals form strong longitudinal in-plane covalent bonds with the \( sp^2 \) orbitals of the neighboring C atoms, called \( \sigma \) bonds as shown in Fig. 2. \( \sigma \) bonds are responsible for rigidity and the robust mechanical properties of graphene. The remaining \( 2p_z \) orbital of each C atom forms lateral covalent bonds with the \( 2p_z \) orbitals of the neighboring atoms called \( \pi \) bonds. Since the \( 2p_z \) orbital of the neighboring C atoms have a small overlap, \( \pi \) bonds are weaker compared to the \( \sigma \) bonds. \( \pi \) bonds are responsible for the electronic properties of graphene.

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The first Brillouin zone of the reciprocal lattice is shown in Fig. 3. The reciprocal space is a convenient concept for describing the reciprocal quantities (usually momentum) in the Fourier domain. It is defined by the reciprocal basis vectors

\[
b_1 = \frac{2\pi}{3a} (\hat{x} + \sqrt{3}\hat{y}) \quad \text{and} \quad b_2 = \frac{2\pi}{3a} (\hat{x} - \sqrt{3}\hat{y}),
\]

which are related to direct vectors through the orthogonal relation \( a_i \cdot b_j = 2\pi \delta_{ij} \). The reciprocal vectors are constructed from the direct space by the relations

\[
b_1 = -2\pi \frac{\hat{z} \times a_2}{\hat{z} \cdot a_1 \times a_2}, \quad b_2 = 2\pi \frac{\hat{z} \times a_1}{\hat{z} \cdot a_1 \times a_2}
\]

The six points at the corners of the Brillouin zone are called the Dirac points. Only two of these points, represented as \( K = \frac{2\pi}{3\sqrt{3}a} (\sqrt{3}\hat{x} + \hat{y}) \) and \( K' = \frac{2\pi}{3\sqrt{3}a} (\sqrt{3}\hat{x} - \hat{y}) \), are inequivalent, and the rest are their periodic images connected by the reciprocal lattice vectors. The nearest neighboring atoms are represented by the vectors \( \delta_1 = \frac{a}{2} (\hat{x} + \sqrt{3}\hat{y}) \), \( \delta_2 = \frac{a}{2} (\hat{x} - \sqrt{3}\hat{y}) \) and \( \delta_3 = -a\hat{x} \).

These vectors are used to derive the tight binding energy band structure in the next section.

C. Energy band structure (tight binding approach)

Tight binding is an efficient method for approximating the band structure, using superposition of a set of basis functions.
closely related to the electronic orbitals of isolated atoms. In this model, it is assumed that the crystal potential only weakly perturbs the electronic orbitals of the isolated atoms. Therefore, the wave function of the electrons in the crystal structure can be approximated by the linear combination of the wave functions of the electrons in the isolated atoms. Since the overlap of the $2p_z$ orbitals in graphene is small, this method provides a good description of the band structure of graphene. The wave function of the electron is then described as the linear combination of the $2p_z$ orbitals over the entire crystal. It may be written as the superposition of the orbitals of the carbon atoms at the $A$ and $B$ sites, namely

$$\psi = c_A \varphi_A + c_B \varphi_B$$

where $\varphi_A$ and $\varphi_B$ are formed, using the Bloch theorem, as the linear combination

$$\varphi_A = \sum_A \exp (\mathbf{j} \cdot \mathbf{r}_A) X (\mathbf{r} - \mathbf{r}_A)$$

$$\varphi_B = \sum_B \exp (\mathbf{j} \cdot \mathbf{r}_B) X (\mathbf{r} - \mathbf{r}_B)$$

where $X (\mathbf{r})$ represents the wave function of the $2p_z$ orbital. Due to the small overlap between the $2p_z$ orbitals between neighboring atoms, the overlap integral is negligible in the tight binding development [17], and may, therefore, be dropped, i.e.

$$\int X (\mathbf{r} - \mathbf{r}_A) X (\mathbf{r} - \mathbf{r}_B) d\mathbf{r} \approx 0$$

The energy states are found by solving the Schrödinger equation

$$\hat{H} \psi = E \psi$$

where $\hat{H}$ is the Hamiltonian of the crystal, which can be written as $\hat{H} = \hat{H}_0 + \Delta U$, with $\hat{H}_0$ representing the Hamiltonian of isolated carbon atoms at their sites and $\Delta U$ representing the perturbation by the neighboring atoms. Replacing $\psi$ in (6) by (2), successively multiplying both sides by $\varphi_A^*$ and $\varphi_B^*$ and integrating the resulting equations over the entire lattice yields [17]

$$H_{AA} c_A + H_{AB} c_B = ESC_A$$

$$H_{BA} c_A + H_{BB} c_B = ESC_B$$

where

$$H_{AA} = \int \varphi_A^* \hat{H} \varphi_A d\mathbf{r}, \quad H_{AB} = H_{BA}^* = \int \varphi_A^* \hat{H} \varphi_B d\mathbf{r},$$

$$H_{BB} = \int \varphi_B^* \hat{H} \varphi_B d\mathbf{r}, \quad S = \int \varphi_A^* \varphi_B d\mathbf{r} = \int \varphi_B^* \varphi_A d\mathbf{r}$$

The energy eigenvalues are found by nullifying the determinant of (7),

$$\begin{vmatrix} H_{AA} - ES & H_{AB} \\ H_{BA} & H_{BB} - ES \end{vmatrix} = 0$$

The assumption (5) leads to $S = N$, where $N$ is the total number of atoms in the crystal, and where the crystal symmetry implies $H_{AA} = H_{BB}$. Therefore (9) becomes

$$E = \frac{1}{N} \left( H_{AA} \pm |H_{AB}| \right) = H_{AA}' \pm |H_{AB}'|$$

with

$$H_{AA}' = \frac{1}{N} \sum_{\mathbf{A}, \mathbf{A}} \exp \left[ j \mathbf{k} \cdot (\mathbf{r}_A - \mathbf{r}_A) \right] X^* (\mathbf{r} - \mathbf{r}_A) \hat{H} X (\mathbf{r} - \mathbf{r}_A) d\mathbf{r}$$

$$H_{AB}' = \frac{1}{N} \sum_{\mathbf{A}, \mathbf{B}} \exp \left[ j \mathbf{k} \cdot (\mathbf{r}_A - \mathbf{r}_B) \right] X^* (\mathbf{r} - \mathbf{r}_A) \hat{H} X (\mathbf{r} - \mathbf{r}_B) d\mathbf{r}$$

Keeping only the nearest neighbour contributions in (11) and (12), $H_{AA}'$ and $H_{AB}'$ are approximated as

$$H_{AA}' = E_0 + \gamma_{AA} \sum_{i=1}^3 \exp (j \mathbf{k} \cdot \mathbf{a}_i)$$

$$= E_0 + \gamma_{AA} \left( 2 \cos (\sqrt{3} R \mathbf{a}) + 4 \cos \left( \frac{\sqrt{3}}{2} R \mathbf{a} \right) \cos \left( \frac{3}{2} R \mathbf{a} \right) \right)$$

$$H_{AB}' = \gamma_{AB} \sum_{i=1}^3 \exp (-j \mathbf{k} \cdot \mathbf{d}_i)$$

$$= \gamma_{AB} \left( \exp (i \mathbf{k} \mathbf{a}) + 2 \cos \left( \frac{\sqrt{3}}{2} R \mathbf{a} \right) \exp \left( -i \frac{1}{2} R \mathbf{a} \right) \right)$$

where $E_0$ is the energy of the electron in the $2p_z$ orbital of an isolated carbon atom [17], $\gamma_{AA}$ is the nearest neighbor hopping energy with an estimated value of 2.8 eV, and $\gamma_{AB}$ is the next nearest neighbor hopping energy with an estimated value of 0.1 eV, which is much smaller compared to $\gamma_{AA}$. The energy band structure $E(k) = E(k) - E_0$ is thus given by


\[ \varepsilon(k) = \pm \gamma_{AB} \sqrt{3 + f(k)} + \gamma_{AA} f(k) \]

\[ f(k) = 2 \cos(\sqrt{3}k_y a) + 4 \cos\left(\frac{\sqrt{3}}{2} k_y a\right) \cos\left(\frac{3}{2} k_x a\right) \]

(14)

[4], plotted in Fig. 4.

Therefore, neglecting the terms with the factor \( \gamma_{AA} \), the band structure at the Dirac points is approximated by the linear symmetric Dirac cone equation

\[ \varepsilon(q) = \pm \hbar v_F |q| \]

(16)

where \( v_F = \frac{3}{2} \gamma_{AB} a \hbar \approx 10^6 \text{ m/s} \) is the Fermi velocity, and \( q \) is the momentum measured with respect to the Dirac points. The striking feature of this energy dispersion function is that the velocity of the carriers is independent of their energy, similar to dispersion of a photon in free space. In this sense, the carriers in graphene are called massless Dirac fermions. Unlike massive particles, whose velocity is increased as the square root of the energy, electrons in graphene are, energy-wise, instantly accelerated to the Fermi velocity, which is the mechanism behind its high mobility. Another interesting property of the Dirac cone is the electron hole energy symmetry, which is the reason for graphene’s ambipolarity. The high symmetry of the band structure also leads to strong interband transition, making graphene a good candidate for light modulators [18-20]. Since graphene has a gapless band structure, the interband conductivity can be easily tuned for terahertz, infrared or optical frequencies by applying the appropriate amount of electrical doping, with applications in tunable graphene-based components such as optical modulators [18-20]. Another interesting property associated with the Dirac cones is the ability to provide unusual conductivities. For some specific parameters, the gapless linear band structure leads to conductivity with a dominant interband contribution. This condition results in a conductivity with a positive imaginary part, as opposed to the Drude conductivity, leading to exotic effects such as TE surface plasmons [5].

**D. Dirac Hamiltonian**

Writing (7) in the following form

\[
\begin{pmatrix}
H'_{AA} & H'_{AB} \\
H'_{BA} & H'_{BB}
\end{pmatrix}
\begin{pmatrix}
\psi_A \\
\psi_B
\end{pmatrix} = E\begin{pmatrix}
\psi_A \\
\psi_B
\end{pmatrix}
\]

(17)

reveals that this equation satisfies the Schroedinger equation with the new 2-spinor wave function
\[ \psi = \begin{pmatrix} c_A \\ c_B \end{pmatrix} \]  
\[ (18) \]

and the effective Hamiltonian
\[ H = \begin{pmatrix} H'_{AA} & H'_{AB} \\ H'_{BA} & H'_{BB} \end{pmatrix} \]
\[ (19) \]

where the two components of the wave function are respectively connected to the A and B sub-lattices through (2). Expanding the effective Hamiltonian (19) around the Dirac point \( K \) results in
\[ \psi(q_x, q_y) = \psi_0(q_x, q_y) \exp\left(-\frac{iq_x B_y + iq_y B_x}{\hbar} \right) \]
\[ \psi_0(q_x, q_y) = \frac{1}{\sqrt{2\pi}} \int d\kappa_x d\kappa_y \psi(K + \kappa_x, K + \kappa_y) \exp\left(-\frac{i\kappa_x q_x + i\kappa_y q_y}{\hbar} \right) \]
\[ (20) \]

The wave function general solution to this equation is
\[ \psi(x, y) = \frac{1}{\sqrt{2\pi}} \int d\kappa_x d\kappa_y \psi(x - \kappa_x, y - \kappa_y) \exp\left(-\frac{i\kappa_x q_x + i\kappa_y q_y}{\hbar} \right) \]
\[ (21) \]

III. GRAPHENE UNDER A STATIC MAGNETIC BIAS

First, graphene’s effective Dirac Hamiltonian is used to describe the quantized energy levels of magnetically biased graphene. Then, based on transitions between these levels, the Kubo conductivity tensor is presented and shown to reduce to a Drude conductivity tensor under conditions that typically prevail in the devices to be described in Section V.

A. Landau levels

Under a magnetic field bias, the carriers are deflected by the Lorentz force and move in cyclotron orbits. The magnetic bias modifies the Hamiltonian of the crystal and limits the allowed cyclotron energies to a set of quantized Landau levels. In conventional parabolic energy dispersions, occurring in conventional two-dimensional electron gas (2DEGs), these levels are equidistant, with an energy difference \( \hbar \omega_c \), where \( \omega_c \) is the cyclotron frequency [21].

For relativistic linear band structures, the quantized allowed energies take a completely different form. Consider the Dirac Hamiltonian for graphene under a normal static magnetic field \( B \). The states near the \( K \) point satisfy the Dirac equation corresponding to (20),
\[ \hbar v_F \sigma \cdot q \psi_K = \epsilon \psi_K \]
\[ (22) \]

or, explicitly,
\[ \hbar v_F \begin{pmatrix} 0 & q_x - q_y \\ q_x + q_y & 0 \end{pmatrix} \begin{pmatrix} \psi_{KA} \\ \psi_{KB} \end{pmatrix} = \epsilon \begin{pmatrix} \psi_{KA} \\ \psi_{KB} \end{pmatrix} \]
\[ (23) \]

Eliminating \( \psi_{KA} \) from the two equations results in
\[ \left( \hbar v_F \right)^2 \begin{pmatrix} 0 & q_x - q_y \\ q_x + q_y & 0 \end{pmatrix} \begin{pmatrix} \psi_{KB} \end{pmatrix} = \epsilon^2 \psi_{KB} \]
\[ (24) \]

The momentum operator is \( \hbar q = -i\hbar \nabla - eA \), where \( A \) is the magnetic vector potential. We will use the Landau gauge for the magnetic vector potential, i.e. \( A = ( -B_y, 0, 0 ) \).

Replacing the momentum operators \( \hbar q_x = -i\hbar \partial_x - eA_x \) and \( \hbar q_y = -i\hbar \partial_y \) reduces (24) to
\[ v_F^2 \left( ( -i\hbar \partial_x + eB_y )^2 + eBh - \hbar^2 \partial_y^2 \right) \psi_{KB} (x, y) = \epsilon^2 \psi_{KB} (x, y) \]
\[ (25) \]

The wave function general solution to this equation is
\[ \psi_{KB} (x, y) = e^{ik_x x} \varphi (y) \]
leading to
\[ \frac{1}{2} \left( \frac{eB}{v_F} \right)^2 \left( y + \frac{\hbar k_y}{eB} \right)^2 - \frac{1}{2} \hbar^2 \partial_y^2 \right) \varphi (y) = \frac{1}{2} \left( \frac{\epsilon}{v_F} \right)^2 - \hbar eB \right) \varphi (y) \]
\[ (26) \]

which is the well-known equation of a quantum harmonic oscillator with frequency \( \omega = |eB| \) and mass \( m = 1 \) [22].

The discrete set of the eigenvalues of (26) are given by
\[ \frac{1}{2} \left( \frac{\epsilon}{v_F} \right)^2 = \hbar \omega \left( n + \frac{1}{2} \right) \]
\[ (27) \]

which, upon solving for \( \epsilon \), finally gives the following set of quantized Landau levels
\[ \epsilon_n = \pm v_F \sqrt{2\hbar |eB| n} \]
\[ (28) \]
where \( n \) is an integer. Therefore, unlike in classical 2DEGs, the quantized Landau energy states in graphene are proportional to the square root of the magnetic field and spaced unequally. As the energies increase, the distance between the Landau levels decreases as \( E_{n+1} - E_n \propto \frac{1}{\sqrt{n}} \).

Consequently, for high enough energies, these levels become very close to each other. Eventually, these discrete energy levels can be treated as a continuum, leading to the simplification of the conductivity models described in the next section.

### B. Conductivity model

Interaction of the carriers in graphene with a tangential electric field is modelled through the conductivity given by the Kubo formula. The Kubo formula describing general many body problems is derived in the context of linear response of the quantum mechanical many body system to a small perturbation [26]-[28]. For an energy-independent response of the quantum mechanical many body system to a very close to each other . Eventually, these discrete energy levels can be treated as a continuum, leading to the simplification of the conductivity models described in the next section.

\[
\sigma_{dc}(\omega, \mu_c, B_0) = \frac{e^2 v_F^2 |eB_0|(\omega - j2\Gamma)\hbar}{\pi} \left[ f_d(M_n) - f_d(-M_n) + f_d(-M_{n+1}) - f_d(M_{n+1}) \right] \frac{1}{\omega - j2\Gamma)^2 \hbar^2} + \frac{\Delta^2}{M_n M_{n+1}} \frac{1}{\omega - j2\Gamma)^2 \hbar^2} \right] \quad (29)
\]

where \( \mu_c \) is the chemical potential, \( \Gamma = 1/2\tau \) with \( \tau \) represents the scattering time, \( f_d = \left(1 + e^{(\epsilon - \mu_c)/k_B \tau} \right)^{-1} \) is the Fermi-Dirac distribution, and \( M_n = \sqrt{\Delta^2 + 2\hbar v_F^2 |eB_0| n} \) is the \( n \)-th Landau level energy. It has been speculated that under very high magnetic fields, a gap might open up in the energy band structure [26]-[28], called an excitonic gap described by the parameter \( \Delta \). For typical values of the magnetic field considered in this paper, \( \Delta \) is set to zero. Equations (29) and (30) involve energy transitions within the conduction, or valence band called the intraband transitions, and energy transitions between valence and conduction bands called interband transitions. For a pedagogical description of these energy transitions the reader is referred to [30]. In the low magnetic field limit \( B_0 \to 0 \), (29) and (30) have a poor convergence. However, since the Landau levels become very dense these series converge into integrals more appropriate for numerical purposes (see [29]).

In the limit of no interband transition, \( \mu_c \gg 2\hbar \omega \), weak Landau quantization (dense Landau levels), \( \mu_c \gg \hbar \omega \), and single carrier type conduction, \( \mu_c \gg k_B T \), the Kubo conductivity reduces to the classical Drude form

\[
\sigma_{dc}(\omega, B_0) = \sigma_0 \frac{1 + j\omega \tau}{(\omega \tau)^2 + (1 + j\omega \tau)^2} \quad (31)
\]

\[
\sigma_0(\omega, B_0) = \sigma_0 \frac{\omega \tau}{(\omega \tau)^2 + (1 + j\omega \tau)^2} \quad (32)
\]

with the DC conductivity

\[
\sigma_0 = \frac{2e^2 \tau}{\pi \hbar^2 k_B T} \ln \left( 2 \cosh \frac{\mu_c}{2k_B T} \right) \quad (33)
\]

The conductivity tensor is thus asymmetric. An important property of an anti-symmetric conductivity tensor is discrimination between right- and left-handed circularly polarized excitations. An anti-symmetric conductivity tensor, \( \sigma = \sigma_d (\mathbf{x} + j\mathbf{y}) + \sigma_o (\mathbf{y} - x\mathbf{y}) \), has two circularly polarized eigenvectors \( \mathbf{x} \pm j\mathbf{y} \) corresponding to the eigenvalues \( \sigma_d \pm j\sigma_o \). The response of such conductivity to a circularly polarized electric field \( \mathbf{E}_{CP} = E_0 (\mathbf{x} \pm j\mathbf{y}) \) can be expressed as
Equation (34) reveals that the conductivities seen by right/left-handed circularly polarized waves are the scalar conductivity \( \sigma \). As a result right- and left-handed circularly polarized waves see different material parameters and behave differently. This process is the cause of nonreciprocity in the magnetoplasmons to be discussed in the next sections.

IV. GRAPHENE MAGNETOPLASMONICS

Surface plasmons are collective oscillations of charges at the interface of metal and dielectric media or a 2DEG, as represented in Fig. 5. The electromagnetic field created in the process exponentially decays from the surface and is thus tightly confined to it. The perturbations produced by the collective oscillation of charges create an electric field in the direction of propagation, while the magnetic field is transverse to the \( \mathbf{k} \) vector. These types of surface waves which propagate in conventional 2DEGs and in graphene are referred to as longitudinal or transverse magnetic (TM) surface plasmons. TM surface plasmons can propagate in materials whose conductivity take a negative imaginary part (assuming the \( \text{e}^{\text{iot}} \) time convention).

Magnetoplasmons are surface charge oscillations in the presence of a static magnetic field. Under the influence of the magnetic field, carriers move in cyclotron orbits and, as a consequence, all the components of the electric and magnetic fields are present. Therefore, magnetoplasmons in infinite graphene are hybrid (TM-TE) modes. The dispersion relation of magnetoplasmons in infinite graphene is obtained by applying electromagnetic boundary conditions on the graphene layer to a surface wave. Assuming an infinite graphene sheet in the \( xz \) plane and a surface wave propagating in the \( z \) direction, with an electric field \( \mathbf{E} = \mathbf{E}_0 \text{e}^{-j\mathbf{k}_0 z - \omega t} \), the dispersion relation reads [8]

\[
\begin{align*}
\frac{k_z^2}{k_0^2} & = 1 + \frac{\alpha_y^2}{\alpha_o^2} + \frac{2s_o^2}{\eta_0^2} \pm \sqrt{\left(1 + \frac{s_o^2}{\eta_0^2} + \frac{s_o^2}{\eta_o^2}\right)^2 - 4s_o^2} - 4s_o^2 \\
\frac{\alpha_y}{k_0} & = -j \frac{1 + s_o^2 + s_o^2 \pm \sqrt{\left(1 + s_o^2 + s_o^2\right)^2 - 4s_o^2}}{2s_o}
\end{align*}
\]

where \( s_o = \eta_o \sigma_o / 2 \), \( s_o = \eta_o \sigma_o / 2 \).

In a graphene strip with a finite width, plasmons or magnetoplasmons exhibit transverse resonances across the strip. The corresponding modes are called the 2D bulk modes of the graphene strip. A magnetically unbiased graphene strip supports an infinite number of bulk modes and two degenerate edge modes, each propagating on a different edge. The dispersion of these modes are shown in Fig. 6 with solid red and blue curves. The dispersion of magnetoplasmons in an infinite graphene sheet with the same material parameters is plotted in a dotted line for comparison.
The edge modes have exponentially decay away from the edge, and therefore have an imaginary transverse wavenumber. Considering the strip dispersion relation, \( k_z^2 = k_0^2 - k_x^2 - k_y^2 = k_0^2 + \alpha_x^2 + \alpha_y^2 \), and taking into account that magnetoplasmons in infinite graphene have exponential decay only along \( y \), the strip modes are expected to have a larger \( k_z \), which explains why they lie beneath the infinite graphene line in Fig. 6. The strip bulk modes have a sinusoidal variation in the \( x \) direction with a real \( k_x \) and therefore, have a smaller \( k_z \) compared to infinite graphene with \( k_x = 0 \), lying above the infinite graphene dispersion curve as expected.

A closer look at the electromagnetic fields of the edge modes in the plane of the graphene strip reveals an interesting feature of their polarization. The electric field inside the plane of the strip for the left and right edges are shown in Fig. 7. The vector field plots show the two edge modes are circularly polarized with opposite (left- and right-handed) polarizations. In the presence of a static magnetic bias, the right-handed and left-handed circularly polarized waves see different conductivities according to (34). Therefore, application of a magnetic bias lifts the degeneracy of the edge modes and confers in them different dispersions. By a similar token, it may be inferred that edge modes of the two edges show asymmetric dispersions for the opposite directions of propagation. This phenomenon is exploited in the next sections in the design of nonreciprocal magnetoplasmonic devices.
Fig. 8. Dispersion curves for magnetoplasmons in a graphene strip with parameters: width $w = 100 \mu m$, relaxation time $\tau = 0.1 \text{ps}$, carrier density $n_e = 10^{13} \text{cm}^{-2}$ and $B_0 = 1 \text{T}$. Edge modes are plotted in red and bulk modes in blue. The dashed curve shows the dispersion for an infinite graphene sheet with the same parameters. The gray area corresponds to the light cone.

One of the consequences of such a nonreciprocity is that the edge modes produce different phase shifts in the forward and backward directions. Assume that the right edge mode is shorted and the left edge mode is excited, as illustrated in Fig. 9. Due to the dispersion asymmetry, the excited edge mode will produce different phase shifts in opposite directions, thus realizing a nonreciprocal phase shifter. The phase shifts, plotted in Fig. 10 for two different carrier densities, show a significant phase difference between the opposite directions.

**B. Nonreciprocal coupler**

The nonreciprocity of edge magnetoplasmons can also be used for the realization of exotic nonreciprocal components, such as nonreciprocal plasmonic couplers. The coupler structure is depicted in Fig. 11 [9]. It consists of two in-plane edge-coupled magnetically biased graphene strips. The coupler is designed such that for the forward direction, the edge mode excited at port 1 couples to port 4. In the backward direction (excitation at port 2) however, coupling does not occur.

To design such a coupler, the physical parameters of the graphene strips should be tuned in such a way that in the forward direction, the edge modes of the adjacent edges of the strips are phase matched and in the opposite direction their phase constants must be unequal. To achieve this condition, the phase constants are adjusted by tuning the carrier density in the strips. The phase constants of the magnetoplasmon modes of the two isolated graphene strips are shown in Fig. 12. The modes of the right strip are shown in solid curves and the modes of the left strip in dashed curves. Red colors represent the edge modes and blue colors the bulk modes.
Thus, when the two strips are close to each other, the mode excited at port 1 couples to port 4. In the backward direction, the situation is different. The modes shown in the highlighted region by ellipses are phase matched. However, they are propagating on the opposite edges, as indicated in Fig. 13 by the green rectangle. The modes propagating on the inner edges, shown in Fig. 13 by the blue rectangle, are phase mismatched, as seen from their dispersions in Fig. 12 indicated by small circles.

Figure 14 shows the simulated electric fields of the coupler in the forward and backward directions. In the forward direction, the edge modes couple and form symmetric and anti-symmetric modes with different phase constants, thus realizing a forward coupler. In the backward direction, however, the edge modes do not couple.

The dispersion curves for the modes of the coupler are plotted in Fig. 15. The symmetrical and anti-symmetrical dispersion curves are plotted in black. The symmetrical and anti-symmetrical modes have different phase constants, and therefore, the edge mode excited at port 1 in the forward direction will couple to port 4.
The vector plot of the transverse electric field for the symmetric and anti-symmetric modes are shown in Fig. 16.

![Fig. 16. Transverse electric field vector plot for the coupled edge modes. (a) anti-symmetric and (b) symmetric.](image)

**C. Plasmonic isolators**

1. Chemically doped isolator

Another interesting nonreciprocal plasmonic mode is the mode propagating along a graphene p-n junction. A chemically doped graphene p-n junction structure is depicted in Fig. 17, consisting of two chemically p-doped and n-doped in-plane graphene strips separated by a nano-gap [12].

![Fig. 17. Chemically doped graphene p-n junction.](image)

The dispersion of the magnetically unbiased structure is shown in Fig. 18. This structure supports two edge modes, a p-n junction mode, shown in red, and an infinite number of bulk modes propagating inside the strips.

![Fig. 18. Slow wave factor and loss for the structure in Fig. 17 with no magnetic bias.](image)

Inspection of the electromagnetic fields in the plane of the strips reveals interesting facts about the p-n junction mode. The electric field of this mode is shown in Fig. 19 for the unbiased structure. In the forward direction, point \( L \) on the left strip sees a right hand circularly polarized (RHCP) wave under propagation, and point \( R \) on the right strip sees a left hand circularly polarized (LHCP) wave. As the magnetic bias is switched on, the conductivity becomes tensorial and RHCP and LHCP waves see equivalent conductivities of \( \sigma_d + j\sigma_a \) and \( \sigma_d - j\sigma_a \), respectively. However, the p and n doped strips will exhibit off-diagonal conductivities with opposite signs \( \sigma_{op} = -\sigma_{on} \). Therefore, in the forward direction, both sides of the junction see the same conductivity, namely \( \sigma_d + j\sigma_a \). Following a similar logic, in the backward direction, the waves on both sides see the same conductivity, namely \( \sigma_d - j\sigma_a \). Therefore, in the presence of a magnetic bias, the p-n junction mode behaves differently in the forward and backward directions. In the forward direction, it becomes concentrated at the junction as the magnetic field is switched on. In the backward direction it splits away from the junction. The dispersions of the
modes and the electric field patterns of the junction mode are plotted in Fig. 20.

![Electric field pattern in the plane of graphene for the structure of Fig. 17 with $\mathbf{B}_0 = 0.2 \text{ T}$. Points R on the right strip sees a clock-wise rotating electric field in the forward direction and a counter clock-wise rotating electric field in the backward direction as the wave propagates. Point L on the left strip sees an oppositely rotating electric field to point R in each direction.](image)

Fig. 19. Electric field pattern in the plane of graphene for the structure of Fig. 17 with $B_0 = 0.2\, \text{T}$. Points R on the right strip sees a clock-wise rotating electric field in the forward direction and a counter clock-wise rotating electric field in the backward direction as the wave propagates. Point L on the left strip sees an oppositely rotating electric field to point R in each direction.

![Slow-wave factor and loss for the magnetoplasmonic isolator in Fig. 17; $w = 100\, \text{mm}$, $t = 0.1\, \text{ps}$, $n = p = 10^{13}\, \text{cm}^{-2}$, $T = 300\, \text{K}$ and $B_0 = 1\, \text{T}$. The p-n junction mode is represented in red.](image)

Fig. 20. Slow-wave factor and loss for the magnetoplasmonic isolator in Fig. 17; $w = 100\, \text{mm}$, $t = 0.1\, \text{ps}$, $n = p = 10^{13}\, \text{cm}^{-2}$, $T = 300\, \text{K}$ and $B_0 = 1\, \text{T}$. The p-n junction mode is represented in red.

The evolution of the p-n junction electric field as the magnetic field is increased is shown in Fig. 21. This nonreciprocal phenomenon can be exploited for the realization of plasmonic isolators. If the magnetically biased structure is excited at the center with the appropriate field distribution, in the forward direction, it will couple to the p-n junction mode. However, in the backward direction, it will be reflected back as there is no mode with energy at the center.

![Evolution of the electric field pattern of the mode propagating at the p-n junction (mode 1) for the forward and backward propagations as the magnetic field is increased.](image)

Fig. 21. Evolution of the electric field pattern of the mode propagating at the p-n junction (mode 1) for the forward and backward propagations as the magnetic field is increased.

2. Electrically doped isolator

One of the benefits of graphene is the possibility of tuning the carrier density through electric biasing. An alternative method for producing p-n junctions in graphene is through the application of a tangential electric field.

![Electrically doped graphene p-n junction.](image)

Fig. 22. Electrically doped graphene p-n junction.

The electric field changes the balance of the carriers on the strip and creates p- and n-doped regions on the right and left sides of the strip. The density of the carriers is found by solving the 2D Poisson equation. The unknown charge density $\rho(x)$ on the strip is determined by solving the integral equation.
where the 2D Green function solution to the Poisson equation is given by
\[
G(x, y; x', y') = -\frac{1}{2\pi\varepsilon_0} \ln \sqrt{(x' - x)^2 + (y' - y)^2}
\]

Finally the non-uniform conductivity across the strip is found by substituting the chemical potential inside the Kubo formula. This magnetically unbiased structure supports a p-n junction mode with maximum energy at the junction, an infinite number of bulk modes, shown in Fig. 25, and two edge modes (not shown here).
VI. CONCLUSIONS

First, the fundamentals of graphene, including carbon bonds, crystallographic characteristics, energy band structure, Dirac Hamiltonian, Landau quantization and conductivity model, was introduced in a self-consistent and pedagogical fashion. Next, graphene magnetoplasmonic principles were overviewed. Finally, the nonreciprocal properties of magnetoplasmonic modes were described and a number of related device applications, such as isolators and couplers, were presented.

REFERENCES


