THE SIMPLEST ANTENNAS: RADIATION, RECEPTION AND SCATTERING BY AN ANTENNA IN 1-DIMENSION

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***Abstract*—This paper aims to provide understanding of scattering and re-radiation by a receiving antenna through a detailed analysis of representatives from the simplest class of possible antennas, 1-dimensional antennas. Complete, exact solutions for the electromagnetic fields associated with such 1-dimensional environment and antenna structures can be obtained using only elementary analysis and functions. These exact solutions may be modeled by transmission line equivalent circuits. The paper carries this development further, reconciling the transmission line models with the traditional antenna equivalent circuit formulation and the scattering matrix representation of antenna theory, in the process explaining the various aspects of antennas: radiation, reception, scattering and re-radiation.**

***Index Terms*—Antenna, Antenna equivalent circuits, Antenna scattering, Scattered power, Scattering parameters, Transmission line models.**

I. INTRODUCTION

This paper aims to provide deeper understanding of antennas, especially receiving antennas, through detailed analysis of representatives of an especially elementary antenna type, 1-dimensional antennas [1]. Commonly, antennas are considered to radiate from some confined volume into (3-dimensional) space. Nevertheless, there are special circumstances, either idealized infinite structures or antennas radiating between parallel conducting plates, when the antenna may be considered as radiating into a 2-dimensional space. This paper analyzes idealized infinite planar geometries in which antennas radiate into a 1-dimensional space.

Construction of a radically simplified structure which nevertheless satisfies all applicable physical principles and constraints for the purpose of developing and illustrating the theory is a time-honored technique. As examples of such constructions one may cite the Carnot engine in Thermodynamics [2] and the Turing machine in Computer Science [3]. In the more modest context of antennas, the 1-dimensional antenna may aspire to a similar role. While 1-dimensional antennas satisfy all applicable physical antenna principles and constraints, exact computation of radiation, reception and scattering requires only elementary algebra and elementary functions. In contrast, analyses of 2- and 3-dimensional antennas involve integral/differential equations and cylindrical or spherical Bessel functions to yield even approximate solutions.

Accordingly, Section II introduces what is perhaps the simplest representative of such 1-dimensional antennas comprised of free space itself and loaded with an infinite plane, thin conducting sheet, *R* (ohms/square). Section III develops the transmission line equivalent circuits for 1-dimensional antennas in general and the preceding simplest representative in particular [4, 5, 6]. Considering reception of a plane wave by this antenna, the exact received and scattered powers are computed. Section IV reinterprets these same results in terms of the traditional antenna equivalent circuit formulation [6 - 10]. Particular attention is given to the role of the radiation resistance which appears in the Thevenin equivalent circuit. A novel analysis of the receive circuit demonstrates the special conditions under which power in the antenna impedance may be given the interpretation of scattered power. Section V presents a scattering matrix formulation specialized for 1-dimensional antennas [11 – 14]. Using this formulation the universal constant relating the gain to the receiving cross-section of a matched 1-dimensional antenna system is established in an especially transparent calculation. Frequently, throughout this paper, we will write the formulas for quantities in several different notations and forms, normalizations, etc.; one notation adapted the approach at hand and equivalents to facilitate comparison with a form obtained directly via a different treatment.

In Section VI, a more complex and interesting class or family of antenna structures is comprised of a dielectric slab transformer loaded with thin resistive sheet [14 – 16]. The dielectric slab transformer is proportioned for zero reflection of a plane wave normally incident on the slab antenna, i.e., zero backscatter. Through this condition the parameters of the slab antenna become functions of the conducting sheet load resistance, in effect defining a different antenna of this class for each value of load resistance. The received power, the power dissipated in the antenna (Thevenin) equivalent circuit resistance, and the scattered power are computed. An error in previous computations of the scattered power by Green [16] and ourselves [1] is corrected. For this class of antennas with zero backscatter, the received power is greater than the scattered power. The scattered power is greater than the power dissipated in the antenna (Thevenin) equivalent circuit radiation resistance.

II. The 1-Dimensional Antenna

The essential feature that distinguishes an antenna among scatterers is the presence of one or more local ports by virtue of which the scattering may be modified or power absorbed by an attached load. The (simplest) antenna shown in Fig. 1 is delimited, so to speak, by the infinite resistive film $R\,(\Omega/square)$ which acts as a load on the local port of this antenna. This rather abstract conception of the simplest antenna will be made clear and concrete by the transmission line model Fig. 2 elaborated in the next section. There the antenna is represented by an ideal, symmetrical 3-port shunt-T junction, transmission lines are attached to the two symmetrical ports, and the resistive load attached at the third port. Incident on the film (from region 1) is a plane wave. In general, a reflection coefficient Γ_1 back into region 1 and a transmission coefficient τ_1 from region 1 into region 2, the shadow region, may be defined.

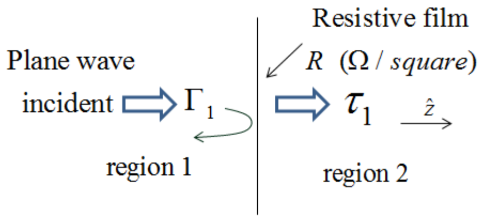

Fig. 1. A 1-dimensional antenna terminated in a plane, infinite resistive film load.

At this point we digress from considerations pertaining to the antenna to insert a brief aside dealing with the vector character of the incident plane wave. It is assumed throughout that the vector polarization of the incident plane wave which propagates on the associated transmission line is matched to the receiving antenna. While, on the one hand, a circuit representation of an arbitrarily polarized incident plane wave does not present any fundamental difficulty, it would detract from the simplicity of presentation, doubling the size of the equivalent circuit (two transmission lines). On the other hand, the amplitude of the matched polarization is easily separated out from an arbitrary incident wave as indicated next.

The electric field vector associated with an arbitrarily polarized incident plane wave $\vec{E}$ may be analyzed as [17]:

$$\vec{E} = \alpha \vec{E}_T^* + \beta \vec{E}_T \times \hat{z} \tag{2.1}$$

where $\vec{E}_T$ is the polarization normally transmitted by the antenna. In the receive mode, the antenna is matched to the vector plane wave with the associated electric field vector $\vec{E}_T^*$, the exact time reverse of the field normally transmitted. In view of the orthogonality relation: $\vec{E}_T \cdot (\vec{E}_T \times \hat{z}) = 0$,

$$\alpha = \frac{\vec{E}_T \cdot \vec{E}}{\left|\vec{E}_T\right|^2}. \tag{2.2}$$

The component $\alpha \vec{E}_T^*$ is obviously independent of the magnitude of $\vec{E}_T$. Accordingly, we will deal with only this polarization matched component of any arbitrarily polarized incident field in the subsequent analysis.

III. Transmission Line Circuit Representation and Analysis of the 1-Dimensional Antenna

The 1-dimensional antenna is amenable to simple formulations using transmission line theory, making use of the well-known correspondence of electric and magnetic field amplitudes of plane waves in space and the voltage and current amplitudes on uniform transmission lines [1, 4]. In view of this correspondence, we obtain the circuit representation shown in Fig. 2 for the physical structure of Fig. 1. The parameters of the equivalent transmission lines are, of course, the propagation constant $k_0 = 2\pi / \lambda_0$ of free space and characteristic impedance R_o of free space. The generator is assumed to have an internal impedance equal to that of free-space R_o. This choice models excitation of a wave amplitude at an infinite remove. The equivalent circuit of this simple antenna is an idealized shunt-T junction, Fig. 2 (and also with matching transformer in Fig. 18.) The resistive sheet is represented by the load at the local port of the antenna, the resistor R.

The incident and reflected waves follow the conventional definitions in terms of rms phasor quantities:

$$E_g = 2\sqrt{R_o}\, a = V + R_o I \tag{3.1a}$$

$$2\sqrt{R_o}\, b = V - R_o I \tag{3.1b}$$

$$\operatorname{Re}\{V I^*\} = |a|^2 - |b|^2. \tag{3.1c}$$

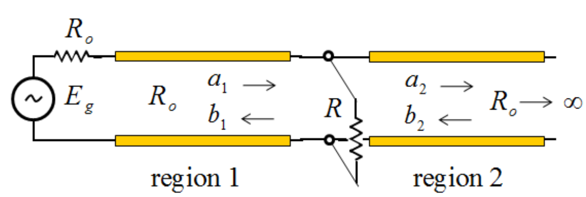

Fig. 2. Transmission line circuit representation, showing an ideal shunt-T load connection.

Equations 3.1 are special cases of the general definition of the normalized voltage scattering parameters further elaborated in Appendix A. Note that transmission line conventions (incidence from left to right) are used here to assign directions to the wave amplitudes a_1 and b_1, a_2 and b_2. The wave parameter on the equivalent transmission line circuit a_1 is proportional to the electric field strength $\alpha \vec{E}_T^*$, (2.1), normalized so that:

$$\left|a_1\right|^2 = \frac{\left|\alpha \vec{E}_T^*\right|^2}{R_o} \quad (Watts / m^2). \tag{3.2}$$

It may be seen from (3.1) the V and I consequently have dimensions of Volts/m and Amps/m respectively. For calculation of powers, Fig. 2 may be reduced to the simple circuit diagram shown in Fig. 3. As both ends of the transmission line are terminated in the characteristic impedance R_o (zero reflection coefficients) the transmission line lengths play no role and are therefore left unspecified. The film resistance is, to a good approximation, independent of frequency.

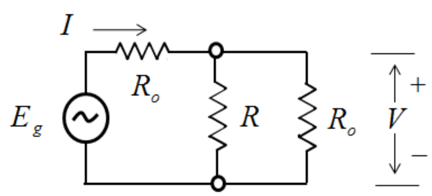

Fig. 3. Reduced circuit representation of Fig. 2.

One may now compute the received and the scattered power. The power received in the sheet (per square meter cross-section of the transversely infinite structure) may be determined from

$$I = \frac{E_g}{R_0}\left(\frac{R+R_0}{R_0+2R}\right) = \frac{E_g}{R_o}\left(\frac{1+\hat{R}}{1+2\hat{R}}\right) \quad \text{where: } \hat{R} \equiv R / R_o \tag{3.3a}$$

$$V = E_g \frac{R}{R_0+2R} = E_g \frac{\hat{R}}{1+2\hat{R}}. \tag{3.3b}$$

Accordingly, the received power P_r is variously given as:

$$P_r = \operatorname{Re}\left\{V I_R^*\right\} \tag{3.4a}$$

$$= V\left(\frac{V}{R}\right)^* = \frac{\left|E_g\right|^2}{R}\left|\frac{\hat{R}}{1+2\hat{R}}\right|^2 \tag{3.4b}$$

$$= \left|a_1\right|^2 \frac{4R_o R}{\left|R_o + 2R\right|^2} = \left|a_1\right|^2 \frac{4\hat{R}}{\left|1+2\hat{R}\right|^2}. \tag{3.4c}$$

The 1-dimensional geometry also allows us to formulate the scattered field and compute "scattered power" in a straightforward manner. The "scattered power" is merely a definite measure of the difference between the total field with the antenna present and the incident field [4,7,10, 11,14,15]. No conservation relation involving received and scattered power exists and none is implied in this computation. The incident field is the field in the absence of the antenna and its (receiver) load (Fig. 2 with the load R removed). Distinguishing the incident and reflected wave quantities of the incident fields within the two regions by zero superscripts

$$a_1^o = a_1 \qquad a_2^o = a_1 \tag{3.5a}$$

$$b_1^o = 0 \qquad b_2^o = 0. \tag{3.5b}$$

In the presence of the antenna and its receiver load, the reflected and transmitted waves are modified so that:

$$a_1 = a_1 \qquad a_2 = (1+\Gamma_1)a_1 \tag{3.6a}$$

$$b_1 = \Gamma_1 a_1 \qquad b_2 = 0 \tag{3.6b}$$

where the reflection and transmission coefficients (for Fig. 2) are given by

$$\Gamma_1 = \frac{R_o R/(R_o+R) - R_o}{R_o R/(R_o+R) + R_o} = \frac{-1}{1+2\hat{R}}, \; -1 \le \Gamma_1 \le 0 \tag{3.7a}$$

$$\tau_1 = (1+\Gamma_1) = \frac{2\hat{R}}{1+2\hat{R}}, \quad 0 \le \tau_1 \le 1. \tag{3.7b}$$

The total field in the presence of the antenna is the sum of the incident and scattered fields. In region 1, the scattered field is therefore found from (3.5) and (3.6)

$$\text{total field in region 1} = a_1 + b_1 = a_1^0 + \text{scattered field in region 1} \tag{3.8a}$$

$$a_1 + \Gamma_1 a_1 = a_1^0 + \text{scattered field in region 1} \tag{3.8b}$$

$$\Gamma_1 a_1 = \text{scattered field in region 1} \tag{3.8c}$$

$$\frac{-a_1}{1+2\hat{R}} = \text{scattered field in region 1.} \tag{3.8d}$$

In region 2, similarly, the scattered field is deduced from (3.5) and (3.6)

$$a_2 + b_2 = a_2^o + \text{scattered field in region 2} \tag{3.9a}$$

$$(1+\Gamma_1)a_1 = a_1 + \text{scattered field in region 2} \tag{3.9b}$$

$$\Gamma_1 a_1 = \text{scattered field in region 2} \tag{3.9c}$$

$$\frac{-a_1}{1+2\hat{R}} = \text{scattered field in region 2.} \tag{3.9d}$$

The results of (3.8) and (3.9) verify an intuitively important aspect. The currents induced in the resistive film radiate (scatter) equally in the two directions. Adding powers scattered into the two directions (regions), the total scattered power is:

$$P_s = 2\left|\Gamma_1 a_1\right|^2 = 2\left|\frac{a_1}{1+2\hat{R}}\right|^2 = \frac{1}{2}\frac{\left|E_g\right|^2 R_0}{\left|R_0+2R\right|^2}. \tag{3.10}$$

The received power and the "scattered power" are equal only when $\hat{R} = 1/2$. This condition corresponds to conjugate match at the receiver load port. Symmetrical scattering and match conditions comport with identification of this simple antenna as a canonical minimum scattering (CMS) antenna

[11]. Indeed, the ideal shunt-T equivalent circuit, Figures 2 and 7, embodies the circuit representation of the CMS antenna concept. The identification is reviewed in the original scattering matrix context in Section V.

The received and scattered powers may be written in terms of the single variable τ_1, $0 \le \tau_1 \le 1$, as:

$$P_r = |a_1|^2 2\, \tau_1\, (1-\tau_1) \tag{3.11}$$

$$P_s = |a_1|^2 2\, (1-\tau_1)^2 . \tag{3.12}$$

Plots of the received and scattered powers are shown in Fig. 4.

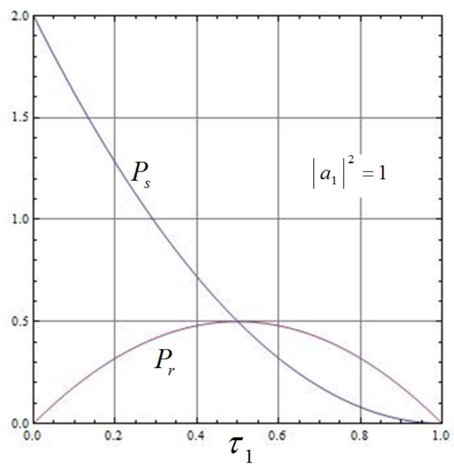

Fig. 4. Received power, P_r and scattered power, P_s versus transmission coefficient, τ_1.

As previously stated, the scattered power (per square meter) is merely one definite measure of the disturbance caused by the antenna, and this measure can easily exceed the power incident (per square meter.) Therefore, not much should be made of comparisons of received and scattered power.

IV. Antenna Circuit Formulation

In contrast with a faithful model such as the transmission line for the 1-dimensional space, and the ideal shunt-T junction for the special simple antenna, engineers regularly describe any antenna by a standard form of equivalent circuit. For the antenna as a transmitter, this standard equivalent circuit is simply the input impedance. For the antenna as a receiver, this standard is a Thevenin (or Norton) equivalent circuit. Relevant properties of the external electromagnetic fields are summarized by the gain (directivity) and receiving cross-section parameters [6, 8, 9.] We begin with some remarks on these circuit descriptions for general antennas before placing the special simple 1-dimensional antenna of the preceding sections into that conventional context.

The standard equivalent circuit of the antenna as a transmitter is shown in Fig. 5. The 1-port antenna itself is represented by its input impedance $Z_a = R_a + jX_a$, where the real part comprises components accounting for (heat) losses and radiation, as indicated in the figure.

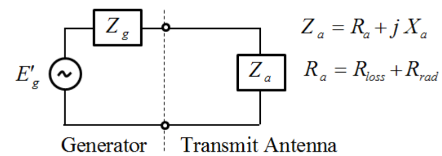

Fig. 5. Antenna as transmitter (excited by generator).

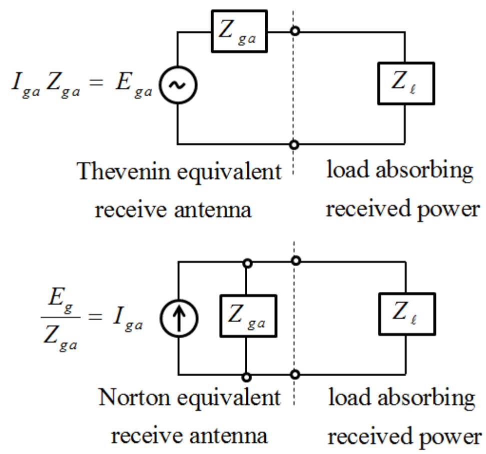

Fig. 6. Thevenin and Norton equivalent receive antenna circuits.

The standard antenna engineer's equivalent circuit of the antenna as a receiver is shown in Fig. 6. The antenna together with the effect of the incident (plane wave) electromagnetic field is represented by either the Thevenin or Norton equivalent circuits at the left of the figure. As is well known, the power apparently dissipated in a Thevenin or Norton equivalent impedance, in general, bears no necessary relation whatever to any actual dissipation mechanism within the physical circuit represented by the Thevenin or Norton circuit. However, the far-field assumptions which underlie the standard antenna formulation result in the circumstance, for reciprocal antennas, that the Thevenin and Norton equivalent impedance equals the input impedance to the antenna, $Z_{ga} = Z_a$. This coincidence has led to various attempts to link the power apparently dissipated in that impedance with power re-radiated and/or scattered by the antenna. Intuitively, it seems clear that the major scattering of a plane wave incident on, say, the back of a parabolic antenna, can have little connection with the input impedance to such an antenna. Indeed, this point was specifically recognized by Silver in his celebrated volume 12 of the MIT Radiation Laboratory Series [6, p. 48], "In general, the power dissipation computed for the equivalent generator impedance is not equal to the power dissipated in the network between $V_G [\equiv E_{ga}]$ and the load; hence, in general it cannot be interpreted as scattered power." This general circumstance will be exemplified by the family of 1-dimensional antennas studied in section VI for which the detailed calculations can be carried out, Fig. 23.

The usually innocuous qualification, "in general" employed in the preceding comments is profoundly significant here. As will be described in the very special case of a canonical minimum-scattering antenna (the simple 1-dimensional antenna analyzed in Section II and Section III being an example of such an antenna), the power dissipated in the radiation resistance power can be linked to the scattered power. A short electric dipole (when considered in isolation from extraneous supporting structures) approximates a theoretical canonical minimum-scattering (CMS) antenna. This provides justification for calculation of scattered power from the power dissipated in the antenna Thevenin equivalent impedance (radiation resistance) appearing in the receive circuit in the case of this very common class of antennas [10].

We will now place the 1-dimensional structure introduced in Section II, Fig. 1, and modeled as transmission line circuit in Section III, Fig. 2, into the present context. As noted previously, the especially simple antenna is modeled by an ideal symmetrical shunt-T junction. The antenna's local port is terminated by the load resistor R representing the physical resistive sheet. The input impedance at this local port (with generator short-circuited) is readily found to be $R_o/2$. Addition of an ideal transformer, turns ratio $1{:}\sqrt{2}$ as in Fig. 7, transforms this input impedance to R_o producing a matched antenna. The conventional transmit equivalent circuit of this antenna is shown in Fig. 8.

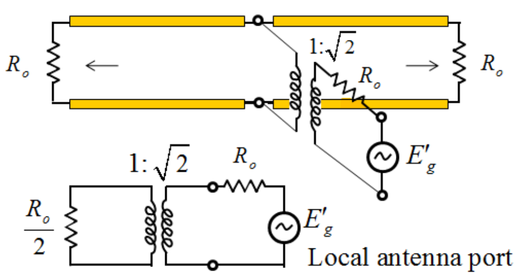

Fig. 7. The simple antenna as transmitter (excited by generator).

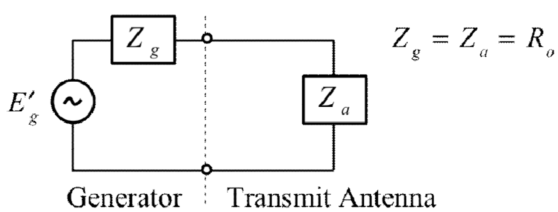

Fig. 8. Conventional equivalent circuit for simple antenna as transmitter (excited by generator).

The parameters for the standard Thevenin receive equivalent circuit for this same antenna are now also easily calculated from Fig. 2. The short circuit input impedance is, of course, R_o, that being the function of the matching transformer. The open circuit voltage is

$$\text{Open circuit local port voltage } = \left(\frac{E_g}{2R_o}\right) R_o\sqrt{2} = \frac{E_g}{\sqrt{2}}. \quad (4.1)$$

The receiving antenna (Thevenin) equivalent circuit is shown in Fig. 9 where $E_g = 2\sqrt{R_0}a_1$ from Fig. 2 and (3.1).

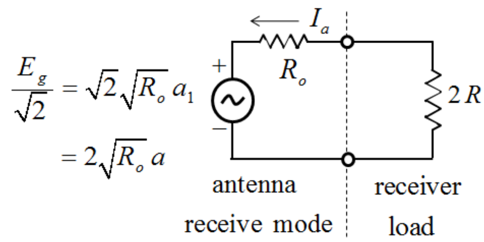

Fig. 9. Conventional Thevenin equivalent circuit for the simple antenna as receiver (Fig. 6) excited by an incident plane wave as in Fig. 2.

Note that, due to the matching transformer, the load resistance equivalent to the resistive sheet becomes $2R$.

The received power, readily calculated from this circuit, is identical to that given in (3.4b)

$$I_a = -I = -\frac{E_g}{\sqrt{2}}\frac{1}{R_0 + 2R} \quad (4.2a)$$

$$P_r = \left|I_a\right|^2 2R = \frac{\left|E_g\right|^2}{2}\left|\frac{1}{R_0 + 2R}\right|^2 2R = \frac{\left|E_g\right|^2}{R}\left|\frac{\hat{R}}{1+2\hat{R}}\right|^2. \quad (4.2b)$$

Substituting for E_g,

$$P_r = \left|a_1\right|^2 \frac{4\hat{R}}{\left|1+2\hat{R}\right|^2}. \quad (4.3)$$

The power dissipated in the antenna resistance is:

$$P_{ant} = \left|I_a\right|^2 R_0 = \frac{\left|E_g\right|^2}{2}\frac{R_0}{\left|R_0 + 2R\right|^2} = 2\left|\frac{a_1}{1+2\hat{R}}\right|^2 = P_s. \quad (4.4)$$

As pointed out, in general, the power apparently dissipated in the impedance of a Thevenin equivalent circuit has no physical interpretation. For the present simple matched antenna which disappears or becomes invisible on open circuit, i.e., as $R \to \infty$, the case of a canonical minimum-scattering antenna, that power equals the scattered power, in agreement with (3.4), (3.10), and (3.12). For more general antennas, as for example the family of antennas discussed in Section VI of this paper, no such equality holds.

Returning to the power received in the load impedance as given by (4.2b), we note that this form is readily transformed in terms of the reflection coefficient ρ (See Appendix A):

$$\rho = \frac{2R - R_o}{2R + R_o} = \frac{2\hat{R} - 1}{2\hat{R} + 1} \quad (4.5a)$$

$$1 - \left|\rho\right|^2 = \frac{8\hat{R}}{\left|1 + 2\hat{R}\right|^2}. \quad (4.5b)$$

We note that the incident wave amplitude in the Thevenin equivalent receive circuit Fig. 9, a is not equal to the space (transmission line) incident wave amplitude a_1; $a = a_1/\sqrt{2}$. Accordingly, the received power may now be expressed as

$$P_r = \frac{\left|a_1\right|^2}{2}\left(1 - \left|\rho\right|^2\right) = P_{inc}\frac{1}{2}\left(1 - \left|\rho\right|^2\right) \quad (4.6)$$

which, on substituting (4.5b), is seen to agree with (4.3).

It is clear that (4.3) and (4.4) imply that the "scattered power" P_s is equal to the received power P_r when the load is matched to the antenna input impedance, $2R = R_0, (\hat{R} = 1/2, \rho = 0)$. This is precisely as would be expected for any canonical minimum-scattering antenna [11] and agrees with the transmission line based calculation of Section III.

The proportionality of antenna gain (directivity) to receiving cross-section for any matched, reciprocal antenna system, the two being related by a universal constant depending only on the dimensionality of the antenna system, is an important elementary result [6, 8, 9]. The specific value of the universal constant is generally found by evaluation of the constant for a particular antenna. For 3-dimensions a short dipole is commonly chosen.

In the transmit mode, the circuit configuration assumes the form shown in Fig. 10. For this simple case, the antenna impedance is the same in the equivalent (actual) circuit for the antenna as transmitter and the equivalent circuit for the receive antenna as shown in Fig. 11.

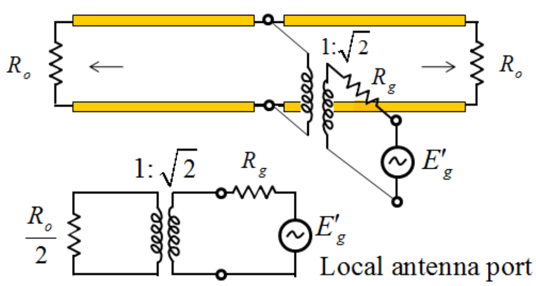

Fig. 10. Transmit antenna transmission line model.

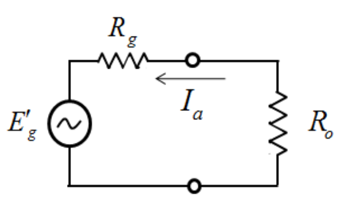

Fig. 11. Circuit representation of the transmit antenna.

The 1-dimensional antenna radiates in only two directions: into region 1 or into region 2. The gain of the antenna radiating into region 1 is written as G(1). Gain is

$$G(\cdot) = \frac{P(\cdot)}{\frac{1}{2}P_{in}}. \qquad (4.7)$$

In view of the symmetry of Fig. 10 and the lossless character of the antenna (transformer and shunt-T junction), we have $P(1) = P(2) = P_{in}/2$. Consequently, $G_o(1) = G_o(2) = 1$; the antenna radiates isotropically. The zero subscript was added to denote the special simple antenna. Detailed circuit calculations of $P(1)$, $P(2)$ and P_{in}, of course, yield the same results.

Given an incoming plane wave, the power received by an antenna is commonly expressed in terms of its receiving cross-section, A where $P_r = A(\cdot)P_{inc}$ and $A(\cdot) = \mathcal{A}(\cdot)\,(1-|\rho|^2)$ and $\mathcal{A}$ is the receiving cross-section of a matched antenna system. It is elementary to show that this receiving cross-section for a matched system is proportional to the gain of the antenna with universal constant of proportionality $\bar{\mathcal{A}}$ [6, 8, 9]. More precisely, there is a specific "universal" constant for antennas radiating and receiving into each: 3-dimensional, 2-dimensional and 1-dimensional space. In summary,

$$A(\cdot) = \mathcal{A}(\cdot)\,(1-|\rho|^2) = \bar{\mathcal{A}}\,G(\cdot)\,(1-|\rho|^2)\,. \qquad (4.8)$$

The formulas for the receiving cross-section in terms of the gain for 3-, 2-, and 1-dimensional antennas are listed in (4.8a, b, c). The constants for 3- and 2-dimensional antennas are well known: the constant $\bar{\mathcal{A}}$ is $\lambda^2/4\pi$ for 3-dimensional antennas and $\lambda/2\pi$ for 2-dimensional antennas. The value of the universal constant for 1-dimensional antennas ½ is inferred from (4.6).

$$P_r = P_{inc}\frac{\lambda^2}{4\pi}G(1-|\rho|^2) = P_{inc}\frac{\lambda^2}{4\pi}(1)(1-|\rho|^2) \qquad (4.8a)$$

$$(isotropic\ 3\text{-}d)$$

$$P_r = P_{inc}\frac{\lambda}{2\pi}G(1-|\rho|^2) = P_{inc}\frac{\lambda}{2\pi}(1)(1-|\rho|^2) \qquad (4.8b)$$

$$(isotropic\ 2\text{-}d)$$

$$P_r = P_{inc}\frac{1}{2}G(1-|\rho|^2) = P_{inc}\frac{1}{2}(1)(1-|\rho|^2). \qquad (4.8c)$$

$$(isotropic\ 1\text{-}d)$$

Equation 4.8c confirms that the universal constant in the formula for receiving cross-section in terms of the gain G for a 1-dimensional antenna is $1/2$. This is in agreement with direct calculation for the isotropic matched antenna terminated in the resistive film (i.e., $R = R_o/2 \rightarrow \rho = 0$). Consistent with this paper $P_{inc} = |a_1|^2$ is defined as a power density W/m^2. Accordingly, P_r will have units of W, W/m, and W/m^2 for the 3-, 2- and 1-dimensional cases, respectively. For the simple matched antenna:

$$P_r\,(\text{matched}) = \frac{|E_g|^2}{4R_o}\frac{1}{2} = |a_1|^2\frac{1}{2} = P_{inc}\frac{1}{2}. \qquad (4.9)$$

Identification of the universal constant for 1-dimensional matched antenna systems is taken up again in Section V. The evaluation will be seen to be particularly direct and transparent in terms of the scattering formulation of antennas.

We remark that a 1-dimensional antenna evidently has a maximum achievable gain of 2.0, realized when all input power is transmitted entirely in one of the two available directions. It follows that a 1-dmensional antenna also has a maximum achievable receiving cross-section, $\bar{\mathcal{A}}\,G(\max) = \frac{1}{2}2 = 1$, absorbing the entire incident power into the antenna load.

The remainder of this section deals with the field scattered by a receiving antenna. In particular, we consider what may be learned regarding the scattered field from the parameters of the conventional equivalent circuit parameters for the receiving antenna. To elicit that information from the equivalent circuit, we make use of the compensation theorems and the principle of superposition.

In brief, the compensation theorems state that, in a circuit any element with a known voltage drop (or carrying a known current value) can be replaced by an ideal, zero internal impedance, compensating voltage source (or an ideal, infinite internal impedance, compensating current source) without affecting the validity of the Kirchhoff circuit equations [4].

Fig. 12 reproduces a simplified version of the conventional equivalent circuit for receive antenna, Fig. 6. Specifically, for CMS antennas, the antenna becomes invisible, does not scatter $P_s \equiv 0$, on open circuit when $R_L \to \infty$.

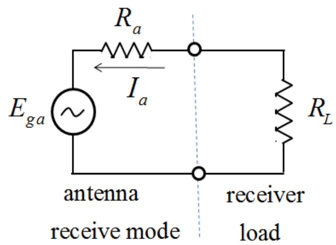

Fig. 12. Thevenin equivalent circuit for a receive antenna.

To show that in the special case of a canonical minimum scattering antenna the power apparently dissipated in the (Thevenin equivalent) resistance of the antenna receive circuit is equal to the scattered power and to indicate the fundamental reason why this is not equal to the scattered power in general, we employ the compensation theorem in the receive circuit Fig. 13 [7,8,17,18]. The load element *R* is replaced by the ideal current generator having magnitude and direction equal to the known current as calculated from the receive circuit. Then, in accordance with the principle of superposition (i.e., independent computations of response of the circuit to the two different generators now present) yields two separate solutions, currents and voltages. The two circuits, each excited by one of the two different generators is show in Fig. 14. The sums (superposition) of these two solutions should, in every instance, yield the total value known from original receive circuit. Since, due to the infinite series resistance associated with the ideal current generator, the first circuit contributes zero current, the sum current obviously has the correct value of the original circuit by construction. The sum of the two voltages V_1 and V_2 is shown to yield the correct value *V*, Eq. (4.10).

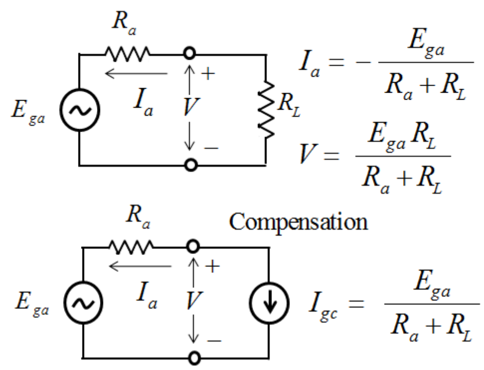

Fig. 13. Compensation theorem applied within the Thevenin equivalent circuit for a receiving antenna.

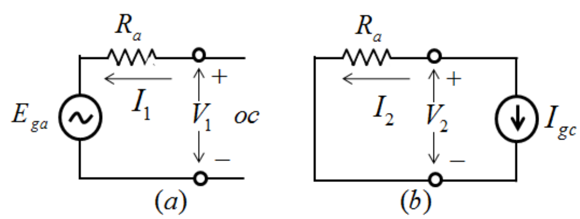

Fig. 14. Principle of superposition of excitations applied to analyze circuit of Fig. 13 into two separate circuits.

From Fig. 14, by construction: $I_a = I_1 + I_2,\ I_1 = 0,\ I_2 = -I_{gc}$. We then compute:

$$V_1 + V_2 = E_{ga} + R_a I_2 = E_{ga} - \frac{E_{ga} R_a}{R_a + R_L} \qquad (4.10a)$$

$$= E_{ga}\left[1 - \frac{R_a}{R_a + R_L}\right] = \frac{E_{ga} R_L}{R_a + R_L} = V \qquad (4.10b)$$

which verifies the correctness of the superposition.

Now consider the scattered fields associated with each of the two circuits of Fig. 14. The minimum scattering antenna disappears, becomes invisible, and does not scatter at all on open circuit, Fig. 14a. The antenna represented in Fig. 14b, the second superposition circuit, is therefore not surrounded by any scattered field (due to 14a) that can interfere with the field radiated by the circuit of the second superposition circuit. Also, that second circuit is identical to the standard equivalent circuit of the CMS antenna as a transmitter since the Thevenin equivalent impedance of the antenna coincides with the input impedance of the antenna. We may therefore identify the radiated (or re-radiated) fields with those that the antenna normally radiates. In the absence of any other external field, the power radiated in the condition of the second superposition equals the power dissipated in the radiation resistance R_a just as is the case when the same antenna is employed as transmitter. Conversely, given a general antenna, such an antenna produces a scattered field even when the general antenna is terminated in an open circuit. That is, the general antenna (including the practically necessary support structures) will produce a scattered field, a so-called structural component, even when there is no (port) current in the receive circuit as in Fig. 14a. The field radiated by the second component, Fig. 14b, is superimposed on this structural component of the field. The two components interfere. Since power depends on the square of the total field, there can be no direct relation of total scattered power to the power that might be radiated (re-radiated) in the absence of the structural component. See Section VI in this connection.

For the special CMS class of antennas, the relationships of received power and scattered power given a real antenna impedance (radiation resistance) and any complex receiver load impedance may be conveniently visualized on the load reflection coefficient plane (Smith chart, impedance normalized to the radiation resistance) Fig. 15 [14]. Loci of constant received power are circles of constant magnitude of reflection coefficient $|\rho| = |\Gamma|$, eqs. (4.6) and (4.8). Note again that these reflection coefficients correspond to incident and reflected wave amplitudes in the circuit of Fig.12; these are distinct from the incident and reflected wave amplitudes on the transmission line equivalent circuit. Loci of constant scattered power are, as will be shown, circles (arcs) centered on the point $\rho = \Gamma = 1$ which also corresponds to infinite load impedance. Indicated values of power, decibels, are relative to the maximum received power when $\rho = \Gamma = 0$.

For the simplest CMS antenna described in Section II and modeled in Section III, points on the real axis correspond to the values graphed in Fig.4. In particular, at the maximum received power point $\rho = \Gamma = 0$, the scattered power equals the received power. When the load impedance is zero, $\rho = \Gamma = -1$, the scattered power is twice the incident power. Relative to the maximum received power (which is half the incident power) this equates to a relative value of 6 dB. In general here, the correspondence of τ_1 with values of Γ is:

$$\tau_1 = 1 - \frac{1}{1 + \left(\frac{2R_L}{R_0}\right)} = \frac{1+\Gamma}{2} \qquad (4.11)$$

where $(2R_L / R_0)$ equals the normalized resistance parameter in the Smith chart.

The construction leading to the loci of constant scattered power is shown in Fig. 16. It follows from the circuit of Fig. 12

$$-\sqrt{R_a}\, I_a = a - b = a(1-\rho) = a(1-\Gamma) \qquad (4.12a)$$

$$P_s = R_a |I_a|^2 = |a|^2 |1-\rho|^2 = |a|^2 |1-\Gamma|^2 \ . \qquad (4.12b)$$

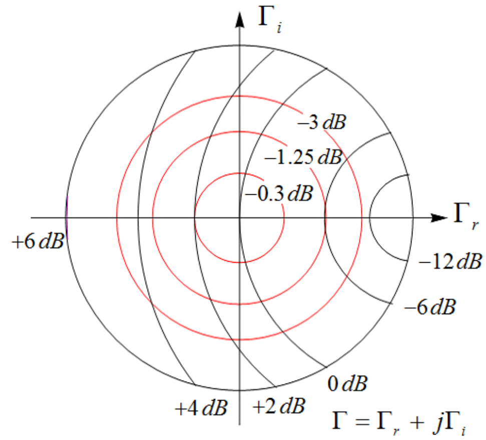

Fig. 15. Load reflection coefficient plane: Red circles – Received power relative to maximum, Black circles – Scattered power relative to maximum received power.

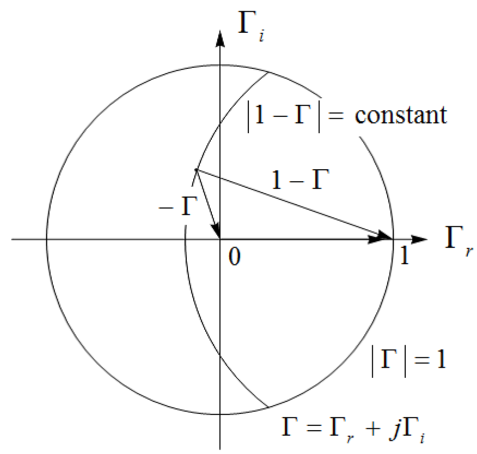

Fig. 16. Construction for scattered power loci (circular arcs) on the complex load reflection coefficient plane.

V. Scattering Matrix Formulation

In general, the scattering matrix of a component (antenna) relates complex wave amplitudes incident on (traveling towards the component) to complex wave amplitudes reflected (traveling away from) the component. Consider now the scattering matrix representation for a 1-port, 1-dimensional antenna [11,12,13]. The conventional notation assigns the letter *a* with subscripts, etc. for the wave amplitudes incident onto the antenna (from either side) and the letter *b* with subscript, etc., for wave amplitudes reflected from the antenna. However here, both to avoid introducing a new notation and at the same time achieve a desirable correspondence with the definition of the wave parameters on the transmission line equivalent circuit, Fig. 2, we make use of the correspondence wave parameter sets whose directions of incidence are defined with respect to either side of a reference plane (transmission line port). The incident wave amplitude a_2, incident in the transmission line sense of a wave traveling towards the right, (in this case from the antenna on the left towards the load to the right) Fig. 2, is intuitively equal to a wave amplitude reflected from the antenna. In the same way, the reflected wave amplitude b_2 (reflected in the transmission line sense of a wave traveling to the left (in this case from the load at the right towards the antenna on the left) Fig. 2, is intuitively equal to a wave amplitude incident on the antenna. When we avail ourselves of these straightforward equivalences, the 1-dimensional antenna with one local port, designated (1), and two-(transmission line)-radiation-ports, designated (2) and (3), is represented as shown in Fig. 17. Radiation port (2) corresponds to transmission line region 1, and radiation port (3) corresponds to transmission line region 2. Note: the intuitive equivalences for the normalized wave parameters used here are rigorously demonstrated in Appendix A. The corresponding scattering matrix representation (5.1) relates column matrices in which the preceding equivalences have been substituted.

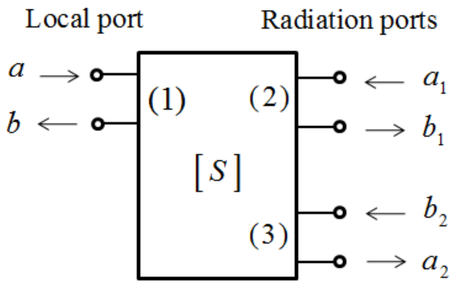

Fig. 17. Scattering matrix of a 1-dimensional antenna.

$$\underline{b}=\begin{bmatrix} b_\alpha \\ b_\beta \end{bmatrix}=\begin{bmatrix} b \\ b_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}\begin{bmatrix} a \\ a_1 \\ b_2 \end{bmatrix}=\begin{bmatrix} S_{\alpha\alpha} & S_{\alpha\beta} \\ S_{\beta\alpha} & S_{\beta\beta} \end{bmatrix}\begin{bmatrix} a_\alpha \\ a_\beta \end{bmatrix}=S\,\underline{a} \tag{5.1}$$

If the antenna satisfies the Lorentz reciprocity constraint, we have $S_{mn} = S_{nm}$.

Accordingly, the gain of the antenna (4.7) can be expressed in terms of the transmission and reflection parameters or, equivalently, in terms of scattering parameters:

$$G(1) = \frac{|b_1|^2}{\frac{1}{2}\left[|a|^2 - |b|^2\right]} \tag{5.2a}$$

$$= \frac{|S_{21}|^2}{\frac{1}{2}\left[1 - |S_{11}|^2\right]} \ . \tag{5.2b}$$

For a matched system, $S_{11} = 0$ and the above expression reduces to

$$G(1) = 2\,|S_{21}|^2 \ . \tag{5.3}$$

Similarly,

$$G(2) = 2\,|S_{31}|^2 \ . \tag{5.4}$$

The corresponding directivities are:

$$D(1) = \frac{|S_{21}|^2}{\frac{1}{2}\left[\,|S_{21}|^2 + |S_{31}|^2\right]} \tag{5.5a}$$

$$D(2) = \frac{|S_{31}|^2}{\frac{1}{2}\left[\,|S_{21}|^2 + |S_{31}|^2\right]} \ . \tag{5.5b}$$

When the antenna is lossless,

$$|S_{21}|^2 + |S_{31}|^2 = 1 - |S_{11}|^2 . \tag{5.6}$$

Then,

$$G(1) = D(1)\ ;\ G(2) = D(2). \tag{5.7}$$

Now consider this same matched reciprocal antenna in the receiving mode. When a wave a_1 is incident (in region 1, port (2)) the power received is:

$$P_r = |S_{12}\,a_1|^2 = |S_{12}|^2 P_{inc} = A(1) P_{inc} \,. \tag{5.8}$$

See (4.8). In view of (5.3) and (5.8), the receiving cross-section for a matched system:

$$A(1) = \bar{\mathcal{Q}}\,G(1) = |S_{12}|^2 = \frac{1}{2}\,G(1) \tag{5.9}$$

since, due to the reciprocity constraint, $|S_{12}|^2 = |S_{21}|^2$.

We conclude that the receiving cross-section of a matched, reciprocal, 1-dimensional antenna system has, quite generally, been shown equal to the gain times a universal constant, $\bar{\mathcal{Q}}$. In the present 1-dimensional case, $\bar{\mathcal{Q}}$ = ½. Indeed, the preceding analysis demystifies once and for all the fundamental reason behind the existence of such a universal constant valid for all matched reciprocal antenna systems of a given dimensionality, c.f., equations (4.8). We note that for 3-dimensional antennas the corresponding demonstration was given by Gately, et al. [12, eq. 17]. However, the necessary notational complexities involving the spherical mode functions obscure its directness and simplicity.

We now develop the scattering matrix of a 1-dimensional canonical minimum-scattering (CMS) antenna. Such a lossless reciprocal antenna is completely defined by its (complex voltage) radiation pattern and the property of becoming invisible when the local receiver port is open circuited, that is, treats or scatters an incident wave exactly as empty, free space.

In many high frequency antennas dissipation within the antenna structure plays a negligible role and the antenna may be idealized as entirely lossless. The normalized scattering matrix of a lossless structure is unitary

$$SS^+ = 1 = S^+ S . \tag{5.10}$$

For a matched antenna system, $S_{\alpha\alpha} = 0$, this constraint becomes

$$\begin{bmatrix} S_{\alpha\beta}S_{\alpha\beta}^+ & S_{\alpha\beta}S_{\beta\beta}^+ \\ S_{\beta\beta}S_{\alpha\beta}^+ & S_{\beta\alpha}S_{\beta\alpha}^+ + S_{\beta\beta}S_{\beta\beta}^+ \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} S_{\beta\alpha}^+S_{\beta\alpha} & S_{\beta\alpha}^+S_{\beta\beta} \\ S_{\beta\beta}^+S_{\beta\alpha} & S_{\alpha\beta}^+S_{\alpha\beta} + S_{\beta\beta}^+S_{\beta\beta} \end{bmatrix}. \tag{5.11}$$

In the notation of the column vectors in (5.1), the scattering matrix relation describing free space is evidently

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}, \tag{5.12a}$$

representing transmission of waves from region 1 into region 2, absent any discontinuity at junction of the transmission line regions. Underlining the absence of an antenna, the full scattering matrix of free space may be written as:

$$F = \left[\begin{array}{c:cc} & & \\ \hdashline & 0 & 1 \\ & 1 & 0 \end{array}\right] = \left[\begin{array}{c:c} & \\ \hdashline & F_{\beta\beta} \end{array}\right] . \tag{5.12b}$$

The above form of the scattering matrix representing 1-dimensional free space as a transparent junction seems intuitively more appealing than the spherical mode representation of free space for a 3-dimensional antenna as a perfect reflector [4,11,12]. However, the difference is more apparent than real. An alternative representation for 1-dimensional free space, which parallels the one necessary in the case of spherical modes, i.e., in terms of symmetrical wave amplitudes converging and diverging from the plane marking the junction of the transmission lines representing region 1 and region 2, is described in Appendix B.

A (matched, lossless) canonical minimum-scattering antenna was defined essentially by the property that such an antenna becomes "invisible" or emulates free space when the accessible local port is open-circuited [11]. Pursuant to this definition, open-circuit conditions (a load with reflection coefficient $\Gamma = 1$) may be imposed at the local port. Consequently, $a = \Gamma b = 1b$, b being the wave incident on the load. Given the load constraint, the local port may be eliminated from the 3×3 antenna scattering matrix

$$a = b = S_{\alpha\beta}\underline{a}_\beta \ . \tag{5.13}$$

So that

$$\underline{b}_\beta = S_{\beta\alpha}a + S_{\beta\beta}\underline{a}_\beta \tag{5.14a}$$

$$\underline{b}_\beta = \left[S_{\beta\alpha}S_{\alpha\beta} + S_{\beta\beta}\right]\underline{a}_\beta . \tag{5.14b}$$

Equating this 2×2 scattering matrix to that of free space

$$F_{\beta\beta} = S_{\beta\alpha}\, S_{\alpha\beta} + S_{\beta\beta} . \tag{5.15}$$

The relation (5.13), emulating free space, fixes $S_{\beta\beta}$ for a given voltage radiation pattern $S_{\beta\alpha}$:

$$S_{\beta\beta} = F_{\beta\beta} - S_{\beta\alpha}\, S_{\alpha\beta} \tag{5.16a}$$

$$S^{+}_{\beta\alpha} S_{\beta\beta} = S^{+}_{\beta\alpha}\, F_{\beta\beta} - S^{+}_{\beta\alpha}\, S_{\beta\alpha}\, S_{\alpha\beta} \tag{5.16b}$$

$$0 = S^{+}_{\beta\alpha}\, F_{\beta\beta} - 1S_{\alpha\beta} \text{ (lossless, unitary condition)} \tag{5.16c}$$

$$S_{\beta\beta} = F_{\beta\beta} - S_{\beta\alpha}S^{+}_{\beta\alpha}F_{\beta\beta} . \tag{5.16d}$$

On the other hand, for a reciprocal antenna, the normalized scattering matrix is symmetrical, the transmit $S_{\beta\alpha}$ and receive $S_{\alpha\beta}$ radiation patterns are connected as

$$S_{\alpha\beta} = S^{T}_{\beta\alpha} . \tag{5.17}$$

The lossless constraint (5.16c) and reciprocity (5.17) must be reconciled by

$$S^{T}_{\beta\alpha} = S^{+}_{\beta\alpha}\, F_{\beta\beta} \tag{5.18}$$

or, in the present 1-dimensional instance

$$\left[S_{21} \quad S_{31}\right] = \left[S^{*}_{31} \quad S^{*}_{21}\right]. \tag{5.19}$$

It follows that the power patterns of 1-dimensional CMS antennas must be symmetrical, that is:

$$\left|S_{21}\right|^2 = \left|S_{31}\right|^2 . \tag{5.20}$$

as is also the case in three dimensions.

As an example illustrating the preceding formulation, consider the case of the simple canonical minimum-scattering antenna, already familiar from Sections II and III, Fig. 18, wherein the resistive film, Fig. 1, the resistive load, Fig. 2, has been replaced by an open-circuit. (Note that the matching transformer, shown explicitly in Fig. 18, merely modifies the value of the resistive load seen at the local port of what was an entirely arbitrary numerical value R in Fig. 2.)

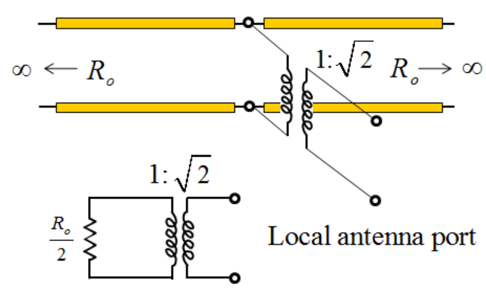

Fig. 18. Transformer necessary to yield matched CMS antenna.

Finally we compute the complete scattering matrix of this antenna and then go on to evaluate what happens to an incident wave $a_1 = 1$ when the antenna is open-circuited at the local port as shown. Of course the latter should show that the structure emulates free space (5.12), i.e., $a_2 = 1$.

We construct the scattering matrix of this lossless reciprocal antenna by (trivially) evaluating the voltage radiation pattern $S_{\beta\alpha}$. Input a at port (1) divides equally between the left and right transmission line regions, circuit ports (2) and (3) by symmetry. The magnitudes follow from the match condition and the conservation of energy

$$S_{\beta\alpha} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = S^{T}_{\alpha\beta} . \tag{5.21}$$

This pattern meets the requirements for a CMS antenna (5.18), $S^{T}_{\beta\alpha} = S^{+}_{\beta\alpha}F_{\beta\beta}$. Constructing the remaining elements of the scattering matrix of such a CMS antenna,

$$S_{\beta\alpha}\, S_{\alpha\beta} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5.22a}$$

$$F_{\beta\beta} - S_{\beta\alpha}\, S_{\alpha\beta} = S_{\beta\beta} = \begin{bmatrix} -\frac{1}{2} & +\frac{1}{2} \\ +\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}. \tag{5.22b}$$

Accordingly, the complete scattering matrix of this simple antenna is

$$S = \frac{1}{\sqrt{2}}\left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & -\frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2} \\ 1 & +\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{array}\right]. \quad (5.23)$$

An incident space wave is now assumed in the left-hand transmission line (region 1, circuit port 2). This incident wave matrix $\underline{a}$ and the resulting reflected wave matrix $\underline{b}$ as computed from this incident wave matrix via the scattering matrix (5.20) are:

$$\underline{a}^T = [0 \quad 1 \quad 0] \qquad \underline{b}^T = \left[\frac{1}{\sqrt{2}} \quad -\frac{1}{2} \quad +\frac{1}{2}\right]. \quad (5.24)$$

But the local port receiving the reflected wave amplitude $b = 1/\sqrt{2}$ is open-circuited, $\Gamma_{loc} = 1$. Consequently, an additional incident wave component $a = \Gamma_{loc} b = 1/\sqrt{2}$ is produced at the local port. This additional incident component produces an additional outgoing reflected wave component, again computed from the scattering matrix (5.20). This is the second column matrix on the left side of (5.21). Adding the two component reflected wave matrices yields the final reflected wave.

It is seen that the radiation components of the total reflected wave column matrix (transmission line wave parameters $\underline{b}_\beta$) are precisely those that would have been obtained in the absence of any antenna. The open circuit condition has rendered the canonical minimum scattering antenna "invisible". That is,

$$\begin{bmatrix} +\frac{1}{\sqrt{2}} \\ \hdashline -\frac{1}{2} \\ +\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \hdashline +\frac{1}{2} \\ +\frac{1}{2} \end{bmatrix} = \begin{bmatrix} +\frac{1}{\sqrt{2}} \\ \hdashline 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ \hdashline b_1 \\ a_2 \end{bmatrix} \quad (5.25)$$

is the total reflected wave, demonstrating $a_2 = 1$.

VI. Application to a Family of Antennas with Zero Backscatter

Having developed various approaches and concepts illustrated by only the simplest example of an antenna, we now seek to apply these to a more general case. A family of 1-dimensional antennas characterized by zero backscatter provides a case of inherent interest; one such family is comprised of a quarter-wave dielectric slab (matching transformer) again terminated by a resistive film load. The geometry of this configuration is shown in Fig. 19.

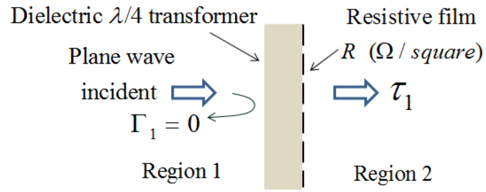

Fig. 19. Resistive film with a quarter-wave dielectric slab.

Equivalent transmission line circuit models for this family of 1-dimensional antennas with resistive sheet load are shown in Fig. 20. In particular, the dielectric constant of the slab, i.e., the transformer turns ratio, will be chosen such that the input impedance R_{in} equals R_o [14, 15, 16]. That is,

$$R_{in} = n^2 \frac{R_o R}{R_o + R} \quad \text{if} \quad n = \sqrt{\frac{R_o + R}{R}} \quad (6.1a)$$

$$R_{in} = R_o \quad (\text{transmission line match}). \quad (6.1b)$$

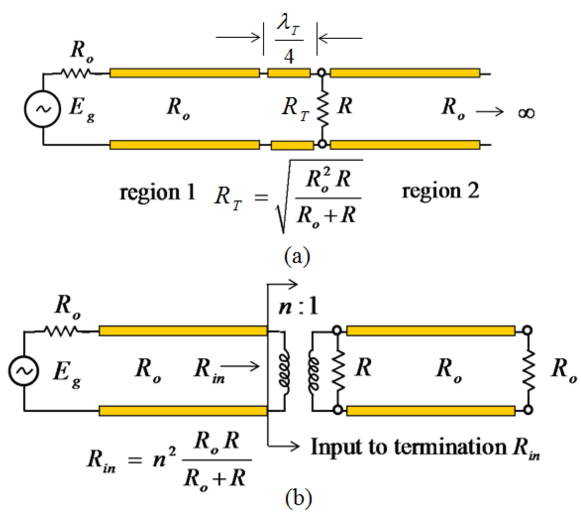

Fig. 20. Circuit models for the zero backscatter antennas - quarter-wave slab transformer with resistive sheet load: a) Faithful transmission line equivalent circuit representation, and b) Ideal transformer equivalent circuit.

As indicated, this defines a family of antenna structures, one member antenna with dielectric constant and slab thickness appropriate for each value of load resistance R. This condition lends a particular interest to this example in that it reduces the antenna reflection coefficient (backscatter) to zero.

Warning! Note that the parameters of the above circuit the load impedance R (corresponding to the selection of one fixed member from the family of zero backscatter antennas) when one investigates, for example, the power received or scattered by this one selected member antenna with some new independently chosen range of load impedances. In those circumstances we shall retain the notation R for the load defining the particular member antenna (n is a function of R) and employ the notation R_L for the independently assigned value of load resistance (n is not a function of R_L) terminating the particular selected antenna.

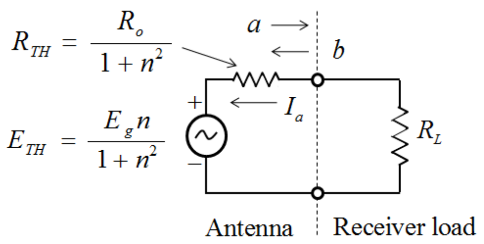

Fig. 21. Standard receive antenna Thevenin equivalent circuit of the zero backscatter antenna.

The standard receive antenna Thevenin equivalent circuit for the family of zero backscatter antennas just described is readily obtained from the circuit Fig. 20a and is shown in Fig. 21. Of course, the backscatter is zero only for the value of $R_L = R$. The internal impedance of the generator is labeled R_{TH} to emphasize its status here. As previously noted for reciprocal antennas, numerically, $R_{TH} = R_A$, the antenna input impedance in the transmit equivalent circuit.

We now compute power received in the load R_L,

$$P_r = \left| \frac{nE_g}{1+n^2} \frac{1}{\frac{R_0}{1+n^2} + R_L} \right|^2 R_L = |a_1|^2 \frac{4R_0 R R_L (R_0 + R)}{|R_0 R + R_0 R_L + 2RR_L|^2}. \quad (6.2)$$

For the special value $R_L = R$, equation 6.2 simplifies to

$$P_r = |a_1|^2 \frac{R_0}{R_0 + R}. \quad (6.3)$$

Bear in mind that a_1 is the incident wave amplitude Fig. 20 and not the incident amplitude a evaluated for the receive circuit Fig. 21. That amplitude a is, from (3.1a):

$$E_{TH} = 2a\sqrt{R_{TH}} = 2a\sqrt{\frac{R_0}{1+n^2}} = \frac{E_g n}{1+n^2}. \quad (6.4)$$

Substituting the values of the Thevenin equivalent parameters and solving for a,

$$a = \frac{E_g n R}{(R_0 + 2R) 2\sqrt{\frac{R_0 R}{R_0 + 2R}}}. \quad (6.5)$$

After some algebra, we obtain the relations

$$|a|^2 = \frac{|E_g|^2}{4R_0} \frac{R_0 + R}{R_0 + 2R} = |a_1|^2 \frac{R_0 + R}{R_0 + 2R}. \quad (6.6)$$

We remark that zero backscatter $R_L = R$ does not imply impedance match (zero reflection coefficient ρ_L) in the standard receive equivalent circuit, Fig. 21. Indeed,

$$\rho_L = \frac{R_L - R_{TH}}{R_L + R_{TH}} = \frac{R_L - \frac{R_0 R}{R_0 + 2R}}{R_L + \frac{R_0 R}{R_0 + 2R}} \quad (6.7)$$

which, when $R_L = R$ simplifies to

$$\rho_L = \frac{R}{R_0 + R} = \frac{1}{n^2}. \quad (6.8)$$

Using this formulation, we check the received power when $R_L = R$:

$$P_r = |a|^2 (1 - |\rho_L|^2) = |a_1|^2 \frac{R_0 + R}{R_0 + 2R} \cdot \frac{R_0 (R_0 + 2R)}{(R_0 + R)^2} = |a_1|^2 \frac{R_0}{R_0 + R}. \quad (6.9)$$

Finally, this received power for $R_L = R$ is also easily checked directly from the transmission line model.

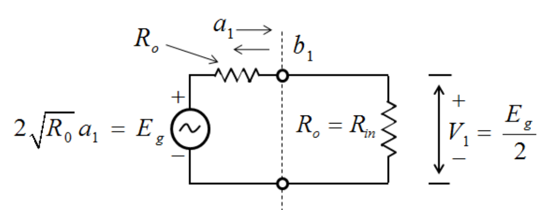

Fig. 22. Equivalent circuit at the input to the transformer when the transformer turns ratio n is chosen to produce a matched load.

By construction, the simple equivalent circuit Fig. 22 is applicable when $R_L = R$. The voltage at the input to the antenna (input to the transformer) is therefore $E_g/2$. The voltage across the load impedance representing the resistive sheet is now found simply by dividing by the turns ratio. The power calculation for the load is then straightforwardly

$$(\text{voltage across } R)\ V_2 = \frac{E_g}{2}\frac{1}{n} = \frac{E_g}{2}\sqrt{\frac{R}{R_o + R}} \quad (6.10a)$$

$$(\text{power into } R)\ P_r = \frac{|V_2|^2}{R} = \frac{|E_g|^2}{4R}\frac{R}{R_o + R} = |a_1|^2 \frac{R_o}{R_o + R} = |a_1|^2 \frac{1}{1 + \hat{R}}. \quad (6.10b)$$

It is clear that as $\hat{R} \to \infty$, no power is delivered to the load. In contrast, as $\hat{R} \to 0$, all of the incident power is delivered to the load; this idealized condition corresponds to $n^2 \to \infty$.

As a check of the actual power balance, we should be able to show that our formulation is consistent with the input power being equal to the power into the load plus the power in region 2 (i.e., the power into the characteristic impedance represented by the infinite transmission line)

$$P_{in} = P_r + |a_2|^2 \quad (6.11)$$

$$|a_1|^2 = |a_1|^2 \frac{1}{1 + \hat{R}} + |a_1|^2 \frac{\hat{R}}{1 + \hat{R}} \quad (6.12)$$

identically.

For convenience and ease of comparison with the work of Green [16], we introduce notation for the (now complex) voltage transmission coefficient τ_1 from region 1 to region 2 in the presence of the antenna

$$a_2 = a_1 \tau_1 \tag{6.13}$$

the reference plane for a_1 is taken at the input to the antenna, i.e., at the input to the quarter wave transformer,

$$\tau_1 = |\tau_1| e^{-j\frac{\pi}{2}} = \frac{1}{n} e^{-j\frac{\pi}{2}} = -j\frac{1}{n} \ . \tag{6.14}$$

Expressed in terms of this transmission coefficient

$$\frac{P_r}{|a_1|^2} = 1 - |\tau_1|^2 . \tag{6.15}$$

Inasmuch as the considerations with the special load $R_L = R$ suffice for our purposes, the obvious algebraic simplification and physical transparency consequent to that choice, we will assume $R_L = R$ in the remainder of this section.

The computation of antenna scattering is based directly on the transmission line equivalent circuits, Figs. 20a and 20b. The scattered field is defined as the difference between the total field and the field in the absence of the antenna.

Absent the antenna is equivalent to $R \longrightarrow \infty$, $n \longrightarrow 1$ (with the consequence $R_T \longrightarrow R_o$ and $\lambda_T \longrightarrow \lambda_o$) in Fig. 20. Quantities representing this incident field will be distinguished with zero superscript. An input reference plane is established at the input to the transmission line section equivalent to the (physical) length of the quarter wave transformer $\lambda_T/4$. Since the dielectric of the transformer has been removed, the wavelength in this length of line reverts to that of free space λ_o with the consequence that the phase length of this section is shortened from the value in the dielectric (which was, of course, $\pi/2$) to

$$\frac{\pi}{2} \cdot \frac{\lambda_T}{\lambda_0} = \frac{\pi}{2} \cdot \frac{1}{n} \ . \tag{6.16}$$

It is important to understand that in the absence of the antenna, the spatial relations, i.e., the physical lengths the antenna occupied must be preserved. This fundamental point, which may not be obvious here in the 1-dimensional case, becomes self-evident in 2- or 3-dimensions. Consider a 3-dimensional spherical dielectric shell as a scatterer. If "absent the scatterer" were interpreted as removal of the shell without preserving the physical space occupied by the shell, the resulting geometry would involve an absurd discontinuity in cross-section with the radial coordinate. Precisely because this point is not as forcefully evident in 1-dimension, it was missed by Green [16]. This paper therefore provides corrected results for the scattered power.

Absent the antenna we have

$$b_1^o = 0 \qquad b_2^o = 0 . \tag{6.17}$$

In the presence of the antenna and its receiver load, Fig. 20a, we have

$$a_1 = a_1 \qquad a_2 = \frac{1}{n} e^{-j\frac{\pi}{2}} \tag{6.18a}$$

$$b_1 = 0 \qquad b_2 = 0 . \tag{6.18b}$$

In region 1, the scattered field is zero as $a_1 = a_1^o$, $b_1 = b_1^o$. In region 2, the total field is:

$$a_2 = a_2^o + \text{scattered field} \tag{6.19a}$$

$$a_1 \frac{1}{n} e^{-j\frac{\pi}{2}} = a_1 e^{-j\frac{\pi}{2}\frac{1}{n}} + \text{scattered field} \tag{6.19b}$$

$$\text{scattered field} = a_1 \frac{1}{n} e^{-j\frac{\pi}{2}} - a_1 e^{-j\frac{\pi}{2}\frac{1}{n}} . \tag{6.19c}$$

The scattered power may therefore (6.14) be written as

$$P_s = |a_1|^2 \left| |\tau_1| - e^{j\frac{\pi}{2}(1-|\tau_1|)} \right|^2 \tag{6.20a}$$

$$\frac{P_s}{|a_1|^2} = \left| |\tau_1| - \cos\frac{\pi}{2}(1-|\tau_1|) - j\sin\frac{\pi}{2}(1-|\tau_1|) \right|^2 \tag{6.20b}$$

$$\frac{P_s}{|a_1|^2} = \left| |\tau_1| - \cos\frac{\pi}{2}(1-|\tau_1|) \right|^2 + \left| \sin\frac{\pi}{2}(1-|\tau_1|) \right|^2 \tag{6.20c}$$

$$\frac{P_s}{|a_1|^2} = 1 + |\tau_1|^2 - 2|\tau_1| \cos\frac{\pi}{2}(1-|\tau_1|) . \tag{6.20d}$$

The incorrect expression for the scattered power previously given by Green and repeated by us in a previous presentation [1], $\left|1-|\tau_1|\right|^2$ lacks the cosine factor.

We now calculate the power apparently dissipated in the Thevenin equivalent resistance in Fig. 21, P_{TH} (which is, of course, also the input resistance of the antenna as a radiator $R_{TH} = R_A$) and compare with the actual value of scattered power just computed from the faithful transmission line model

$$P_{TH} = \left| \frac{E_g n}{1+n^2} \frac{1}{\frac{R_0}{1+n^2} + R} \right|^2 \frac{R_0}{1+n^2} \tag{6.21a}$$

$$P_{TH} = \frac{|E_g|^2}{4R_o} \frac{R_o}{R_o + R} \frac{1}{R} \frac{R_o R}{R_o + 2R} = |a_1|^2 \frac{R_o}{(R_o + R)} \frac{R_o}{(R_o + 2R)} \ . \tag{6.21b}$$

In terms of the variable τ_1, we find

$$P_{TH} = |a_1|^2 \left(1 - |\tau_1|^2\right)^2 \Big/ \left(1 + |\tau_1|^2\right); \qquad (6.22)$$

$$P_{TH} \neq P_s \;. \qquad (6.23)$$

We are not surprised by this circumstance that is entirely in line with Silver's warning on this score and cited earlier in this paper. As pointed out in Section III, agreement of scattered power with the power apparently dissipated in the Thevenin equivalent impedance may be justified only for CMS antennas. Only for such antennas that do not scatter at all on open circuit, does the Thevenin equivalent resistance in the receive circuit provide a basis for calculating the scattered power. On open circuit, the present antennas reduce to dielectric slabs. The scattering from these dielectric slabs is evaluated in Appendix C. There is, in general, no physical interpretation for the scattered power calculated in (6.17).

Normalized plots of P_s, P_r, and P_{TH} are presented in Fig. 23. It is seen that the scattered power is not necessarily greater or equal to the received power. Indeed, it is less than the received power for the zero backscatter case. Nor is either power (for these more general, not CMS antennas) equal to the power apparently dissipated in the Thevenin equivalent antenna impedance.

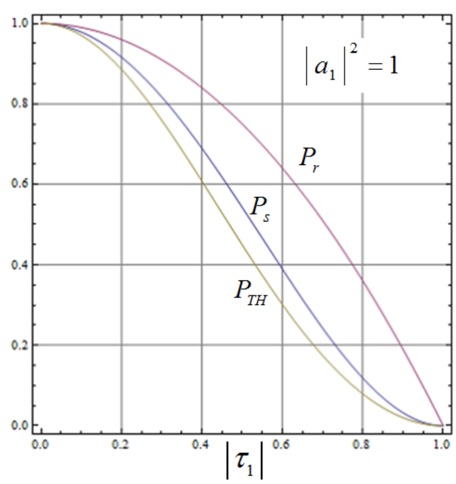

Fig. 23. Plot of normalized powers for the family of reflectionless antennas: received power P_r, scattered power P_s, and power apparently dissipated in the Thevenin equivalent impedance $R_{TH} = R_A$ of the standard receive circuit P_{TH}.

We now consider the present family of antennas as transmitters. The transmission line geometry is given in Fig. 24 while the standard antenna equivalent circuit is given in Fig. 25.

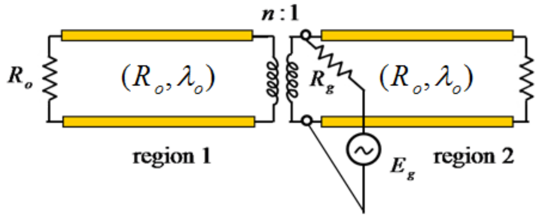

Fig. 24. Ideal transformer transmission line representation for the zero backscatter antenna in the transmit mode.

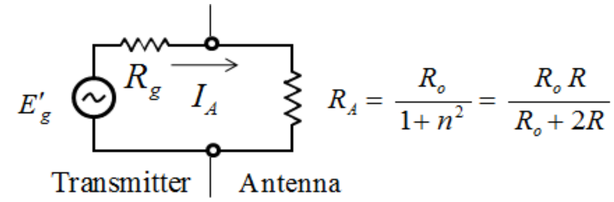

Fig. 25. Standard equivalent circuit for the zero backscatter antenna transmit mode; note $R_A = R_{TH}$.

We wish to compute the gain of the antenna. This parameter is independent of the internal impedance of the antenna system $R_g = R_A$ as indicated in Fig. 25. The total radiated power is

$$\begin{aligned} P_t &= |I_A|^2 R_A = \frac{|E'_g|^2}{4R_A} = |a|^2 \\ &= \frac{|E'_g|^2}{4}\frac{(R_o + 2R)}{R_o R} = \frac{|E'_g|^2}{4}\frac{(1+n^2)}{R_o}. \end{aligned} \qquad (6.24)$$

As a check, the same total power is found by adding up the power into each region of Fig. 24.

$$P(1) = \left|\frac{E'_g}{2}\right|^2 \frac{n^2}{R_o} \;; \quad P(2) = \left|\frac{E'_g}{2}\right|^2 \frac{1}{R_o} \,. \qquad (6.25a)$$

$$P_T = P(1) + P(2) = \frac{|E'_g|^2}{4}\frac{(1+n^2)}{R_o}. \qquad (6.25b)$$

Having these separate power calculations permits calculation of the gain (equal to the directivity for lossless antennas) into each region (1) and (2) :

$$D(1) = \frac{P(1)}{\left(\frac{P_T}{2}\right)} = \frac{\left|\frac{E'_g}{2}\right|^2 \frac{n^2}{R_o}}{\frac{1}{2}\frac{|E'_g|^2}{4}\frac{(1+n^2)}{R_o}} \qquad (6.26a)$$

$$= \frac{2\,n^2}{1+n^2} = \frac{2\,(R_o + R)}{(R_o + 2R)} = G(1) \;; \qquad (6.26b)$$

$$D(2) = \frac{P(2)}{\left(\frac{P_T}{2}\right)} = \frac{\left|\frac{E'_g}{2}\right|^2 \frac{1}{R_o}}{\frac{1}{2}\frac{|E'_g|^2}{4}\frac{(1+n^2)}{R_o}} \qquad (6.27a)$$

$$= \frac{2}{1+n^2} = \frac{2R}{(R_o + 2R)} = G(2)\,. \qquad (6.27b)$$

These values of gain or directivity can now be turned to compute received power in the conventional way from the receiving cross-section. The receiving cross-section for a matched antenna system $\mathcal{A}$ is proportional to the gain (i.e., equal to the gain times a universal constant $\bar{\mathcal{A}}$). As previously indicated, equations (4.8), the received power should equal the incident power multiplied by the receiving cross-section. It must be recalled that a zero back-scatter antenna does not form a matched antenna, ρ is given by (6.3). We therefore have, with some algebra,

$$P_r = P_{inc}\,\mathcal{Q}\,(1-|\rho|^2) = |a_1|^2\,\bar{\mathcal{Q}}\,D(1)\,(1-|\rho|^2) \quad (6.28a)$$

$$P_r = |a_1|^2 \frac{1}{2}\frac{2n^2}{1+n^2}(1-|\rho|^2) = |a_1|^2 \frac{R_o}{R_o+R}. \quad (6.28b)$$

Comparison with (4.8c) verifies that $\bar{\mathcal{Q}}$ must have the value 1/2 substituted in (6.28b) as found in Sections III and IV.

VII. Conclusions

This paper has provided a new path to deeper understanding of receive antennas through detailed analysis of an especially elementary antenna type, a 1-dimensional antenna. Analyses of idealized infinite planar geometries in which antennas radiate into a 1-dimensional space were presented. While computed results, of course, do not directly carry over to 2- and 3-dimensional antennas, they can provide a template for corresponding calculations in these functionally more complicated domains. The paper focused on the details of radiation, reception, scattering and re-radiation in terms of various antenna (circuit) formulations satisfying all applicable physical constraints. The understanding gained in this way is consequently applicable to antennas in general. Of particular importance was the analysis of the relation of scattered power to the power into the antenna impedance. Because of the 1-dimensional radiation assumption and the inherent ability to calculate received and scattered power from a transmission line perspective, it is easy to verify the special case, the class of CMS-like antennas, in which interpreting the power into the antenna's radiation resistance correctly (or very nearly) yields the scattered power, as well as to demonstrate the fallacy of assuming that this calculation always yields the correct value of scattered power for antennas in general.

Appendix A - The Scattering Parameters

The definition of transmission and reflection coefficients used in this paper follows the transmission line convention as shown in Fig. A1. The subscripts r and ℓ indicate "a" parameter incidence from the right and left, respectively.

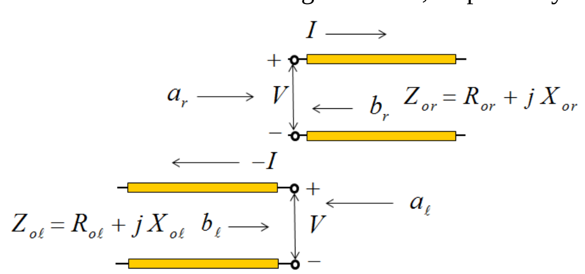

Fig. A1. Transmission line convention.

$$2\sqrt{R_{or}}\,a_r = V + Z_{or} I \quad (A1a)$$

$$2\sqrt{R_{or}}\,b_r = V - Z_{or}^* I \quad (A1b)$$

$$2\sqrt{R_{o\ell}}\,a_\ell = V + Z_{o\ell}(-I) \quad (A2a)$$

$$2\sqrt{R_{o\ell}}\,b_\ell = V - Z_{o\ell}^*(-I). \quad (A2b)$$

If $Z_{o\ell} = Z_{or}^*$, then:

$$a_\ell = b_r \quad (A3a)$$

$$b_\ell = a_r. \quad (A3b)$$

Z_{or} and $Z_{o\ell}$ are normalization numbers and are not necessarily characteristic impedances [20].

The power delivered to the right is found as follows:

$$\left|2\sqrt{R_{or}}\,a_r\right|^2 = \left|V + Z_{or} I\right|^2 \quad (A4a)$$

$$\left|2\sqrt{R_{or}}\,b_r\right|^2 = \left|V - Z_{or}^* I\right|^2. \quad (A4b)$$

$$4R_{or}|a_r|^2 = |V|^2 + V Z_{or}^* I^* + Z_{or} I V^* + |Z_{or} I|^2 \quad (A5a)$$

$$4R_{or}|b_r|^2 = |V|^2 - V Z_{or} I^* - Z_{or}^* I V^* + |Z_{or} I|^2. \quad (A5b)$$

Subtracting (A5b) from (A5a) gives the power delivered to the right:

$$4R_{or}\left[|a_r|^2 - |b_r|^2\right] = (Z_{or}^* + Z_{or}) V I^* + (Z_{or} + Z_{or}^*) I V^* \quad (A6a)$$

$$4R_{or}\left[|a_r|^2 - |b_r|^2\right] = 2R_{or} V I^* + 2R_{or} I V^* \quad (A6b)$$

$$|a_r|^2 - |b_r|^2 = \mathrm{Re}\{V I^*\}. \quad (A6c)$$

Similarly, the power delivered in terms of left incidence parameters:

$$|a_\ell|^2 - |b_\ell|^2 = \mathrm{Re}\{V(-I)^*\} = -\mathrm{Re}\{V I^*\}. \quad (A7)$$

The corresponding reflection coefficients become:

$$\rho_r = \frac{b_r}{a_r} = \frac{V - Z_{or}^* I}{V + Z_{or} I} = \frac{Z - Z_{or}^*}{Z + Z_{or}}; \quad (A8a)$$

$$\rho_\ell = \frac{b_\ell}{a_\ell} = \frac{V - Z_{ol}^*(-I)}{V + Z_{ol}(-I)} = \frac{V + Z_{or} I}{V - Z_{or}^* I} = \frac{a_r}{b_r}; \quad (A8b)$$

$$\rho_r = \frac{1}{\rho_\ell}. \quad (A8c)$$

The above reflection coefficient defined with reference to the normalized voltage scattering amplitudes ρ_r differs theoretically from the reflection coefficient Γ commonly defined exclusively with reference to transmission line wave amplitudes. In general, the voltage and currents on a uniform transmission line may be written [4]

$$V(z) = Ae^{-\gamma(z-z_0)} + Be^{+\gamma(z-z_0)} = Ae^{-\gamma(z-z_0)}\left[1+\Gamma(z)\right] \quad (A9a)$$

$$Z_0 I(z) = Ae^{-\gamma(z-z_0)} - Be^{+\gamma(z-z_0)} = Ae^{-\gamma(z-z_0)}\left[1-\Gamma(z)\right] \quad (A9b)$$

where

$$\Gamma(z) = \Gamma(z_0)e^{+2\gamma(z-z_0)}, \qquad \Gamma(z_0) = \frac{B}{A}. \quad (A10)$$

Dividing, we obtain

$$\frac{V(z)}{Z_0 I(z)} = \frac{Z}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)}. \quad (A11)$$

Solving this relation for Γ and dropping the coordinate argument,

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}. \tag{A12}$$

Formally, the expressions for the two reflection coefficients differ only in the complex conjugate appearing in the numerator of the formula for ρ_r , (A8a).

For real values of $Z_o = R_o$, $\mathrm{Im}\{ Z_o \} = 0$, the two reflection coefficients are numerically equal. Under the same constraint, we have at any point on a transmission line.

$$V = \sqrt{R_0}\,[a+b] = A + B \tag{A13}$$

so that the normalized voltage scattering amplitudes and transmission line voltage wave amplitudes differ only by a constant factor of $\sqrt{R_0}$.

Appendix B - The Scattering Parameters in Alternative Coordinates

In the presentation of the scattering matrix formulation for a 1-dimensional antenna, Section V, we were at pains to adopt a formulation which connected smoothly with the transmission line model and circuit formulations of the preceding sections. However, one consequence of this approach was to obscure the equivalence of the 1-dimensional results with those obtained for 3-dimensional antennas employing the spherical vector mode basis functions. This appendix clarifies the relationship between the preceding forms, in particular the form of the matrix representing free space, with that found for the corresponding scattering matrix formalism as developed for antennas in 3-dimensions [11, 12, 13].

As indicated, the basis for the scattering representation in 3-dimension are incoming (symmetrically converging) and outgoing (symmetrically diverging) spherical wave modes. These waves propagate in the $\hat{r}$ direction, functionally as specified by the appropriate spherical Hankel function, in all directions from the origin of coordinates. In free space (no antenna structure) the origin must be an ordinary non-singular point. This regularity requirement forces the equality of incoming and outgoing amplitudes, i.e., the scattering matrix is necessarily the unit matrix. Similar considerations govern in 2-dimensions. However, for plane waves, the absence of singularities leads to flexibility in the scattering representation of free space and in particular to the form shown in (5.12). An alternative basis for the waves on our transmission lines, one that mimics the symmetry of waves incoming and outgoing from a central point, does indeed closely parallel the 3- and 2-dimensional forms.

Consider the alternative coordinates induced by the real orthogonal transformation in the matrix sub-space of the radiation ports (transmission line) wave parameters:

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{B1a}$$

$$\begin{bmatrix} a_1 \\ b_2 \end{bmatrix} = T \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} \quad ; \quad \begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = T \begin{bmatrix} b_1' \\ b_2' \end{bmatrix}. \tag{B1b}$$

The scattering matrix of free-space becomes:

$$F'_{\beta\beta} = T^{-1} F_{\beta\beta} T = T^T F_{\beta\beta} T \tag{B2a}$$

$$F'_{\beta\beta} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{B2b}$$

The new coordinate system implies symmetric and anti-symmetric excitations (radiation and reflection) from a plane in free space. The resulting scattering matrix representing free space is now diagonal with unit reflection entries.

Appendix C – 1-Dimensional Scattering by a Quarter-Wave Dielectric Slab

A quarter-wave dielectric slab remains when the resistive film load on a member of the family of zero-backscatter antennas of Section VI is removed, $R_L \to \infty$, effectively open-circuiting the antenna, Fig. C1. As we have explained, the fields scattered by the slab interfere with fields re-radiated by the receive antenna equipped with its normal load R so that the total scattered power is generally not equal to the power apparently lost in the radiation resistance of the Thevenin equivalent receive circuit. We shall present two calculations of this scattered power. The first follows the same outline as used in Section VI to compute the scattered power from the receiving antenna; the second will employ the alternative symmetrical mode set of Appendix B.

As established in Section VI, it is key to maintain the same spatial relations of reference planes for incident and reflected wave parameters in the absence of the slab as exists when the slab is present. Thus, for each member of the family of zero backscatter antennas, a free-space length equal to a quarter wavelength in the appropriate transformer dielectric must be maintained when the slab is removed.

In region 1

$$\textit{Total Field} = \textit{Incident Field} + \textit{Scattered Field} \tag{C1}$$

$$a_1 + b_1 = a_1^0 + b_1^0 + \textit{Scattered Field} \tag{C2}$$

$$a_1 + a_1 |\Gamma_S| e^{-j\pi} = a_1 + 0 + \textit{Scattered Field} \tag{C3}$$

$$a_1 |\Gamma_S| e^{-j\pi} = \textit{Scattered Field} \tag{C4}$$

$$a_1 \frac{n^2-1}{n^2+1} e^{-j\pi} = \textit{Scattered Field}\,. \tag{C5}$$

The scattered power in region 1 is therefore

$$P_{S1} = |a_1|^2 \left| \frac{n^2-1}{n^2+1} \right|^2. \tag{C6}$$

Here, the phase shift of the reflection coefficient of the transformer through the quarter wave dielectric slab to the established input reference plane played no effective role, since $b_1^0 = 0$.

In region 2

$$\textit{Total Field} = \textit{Incident Field} + \textit{Scattered Field} \tag{C7}$$

$$a_2 + b_2 = a_2^0 + b_2^0 + \textit{Scattered Field} \tag{C8}$$

$$a_1 e^{-j\frac{\pi}{2}}|\tau_S| + 0 \;=\; a_1 e^{-j\frac{\pi}{2}\frac{1}{n}} + 0 + \textit{Scattered Field} \qquad \text{(C9)}$$

$$a_1 e^{-j\frac{\pi}{2}}\left[|\tau_S| - e^{+j\frac{\pi}{2}\left(1-\frac{1}{n}\right)}\right] = \textit{Scattered Field} \qquad \text{(C10)}$$

$$a_1 e^{-j\frac{\pi}{2}}\left[\frac{2n}{n^2+1} - \cos\frac{\pi}{2}\left(1-\frac{1}{n}\right) - j\sin\frac{\pi}{2}\left(1-\frac{1}{n}\right)\right] = \textit{Scattered Field} \qquad \text{(C11)}$$

The scattered power in region 2 is therefore

$$P_{S2} = |a_1|^2 \left|\frac{2n}{n^2+1} - \cos\frac{\pi}{2}\left(1-\frac{1}{n}\right) - j\sin\frac{\pi}{2}\left(1-\frac{1}{n}\right)\right|^2 \qquad \text{(C12)}$$

$$P_{S2} = |a_1|^2 \left[\left(\frac{2n}{1+n^2}\right)^2 - \frac{4n}{1+n^2}\cos\frac{\pi}{2}\left(1-\frac{1}{n}\right) + \right. .$$

$$\left. \left(\cos\frac{\pi}{2}\left(1-\frac{1}{n}\right)\right)^2 + \left(\sin\frac{\pi}{2}\left(1-\frac{1}{n}\right)\right)^2\right] \;. \qquad \text{(C13)}$$

The total scattered power from the slab is therefore

$$P_s^S = P_{s1}^S + P_{s2}^S \qquad \text{(C14)}$$

$$\frac{P_s^S}{|a_1|^2} = \left(\frac{n^2-1}{n^2+1}\right)^2 + \left(\frac{2n}{n^2+1}\right)^2 - \frac{4n}{1+n^2}\cos\frac{\pi}{2}\left(1-\frac{1}{n}\right) + 1 \qquad \text{(C15)}$$

$$\frac{P_s^S}{|a_1|^2} = 2 - \frac{4n}{1+n^2}\cos\frac{\pi}{2}\left(1-\frac{1}{n}\right) \;. \qquad \text{(C16)}$$

The last simplification follows from the unitary character of the scattering matrix of the lossless ideal transformer; the sum of the squares of reflection and transmission coefficients is unity. If, furthermore, we make use of the relation between the turns ratio n and the transmission coefficient for the family of zero-backscatter antennas $|\tau_1| = 1/n$, we can readily plot the scattered power from the slab on the same abscissa used for the scattered power from the zero-backscatter antennas, Fig. 23. See Fig. C3.

We now compute the scattered power from the quarter wave slab employing the alternative symmetrical modes, introduced in Appendix B, which more closely resemble the incoming and outgoing waves naturally associated with the central location of a 2- or 3-dimensional antenna.

The symmetrical mode amplitudes corresponding to an incident plane wave amplitude a_1 follow from equations (B1). Since T is a real (unitary) orthogonal transformation for which, in particular $T = T^+ = T^{-1}$, all power relations will be preserved;

$$\begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ 0 \end{bmatrix} \;. \qquad \text{(C17)}$$

As the dielectric slab (without conducting film load) is a symmetrical structure, we choose the middle of the slab as the plane of symmetry for the symmetrical modes, Fig. C2a. Furthermore, because the parameters are *normalized voltage wave parameters* (proportional to transverse electric field), it follows that the symmetrical mode satisfies open circuit (perfect magnetic conductor) conditions on the symmetry plane while the antisymmetric mode satisfies short circuit (perfect electric conductor) conditions on the symmetry plane. [4] Given these conditions at the middle plane of the structure and the symmetrical equivalent circuit, Figs. C2, we readily infer the corresponding reflection coefficients at the input of the slab.

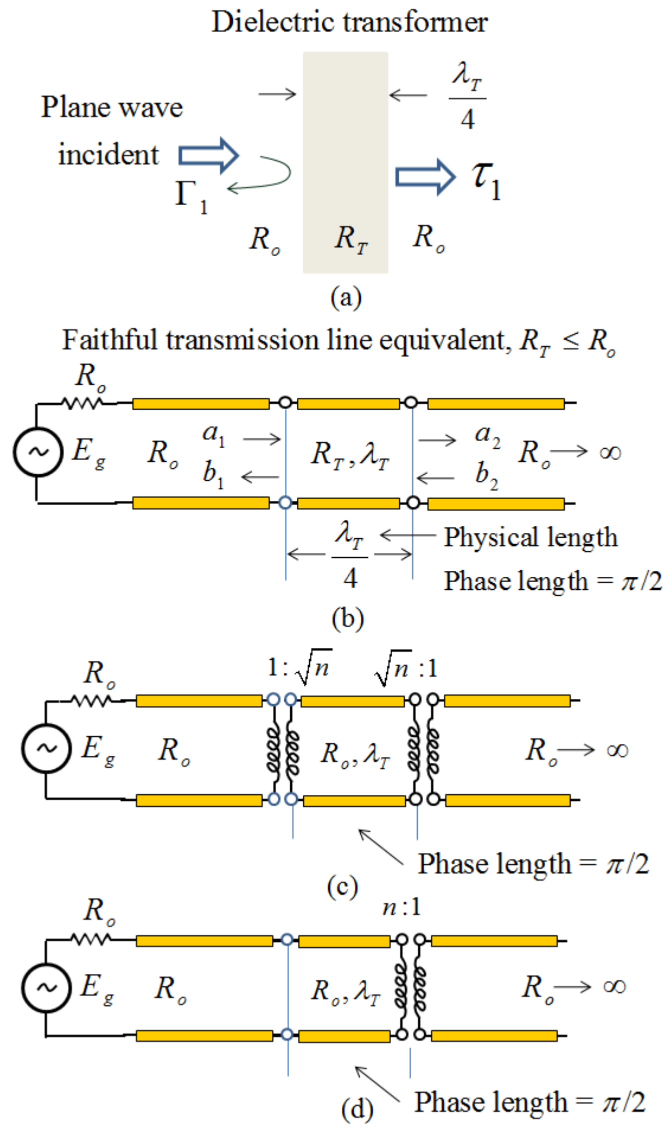

Fig. C1. Quarter wave dielectric slab (remaining when zero backscatter antenna is open-circuited, i. e., resistive film load removed) and equivalent circuit representations: a) Physical structure, b) Faithful transmission line circuit representation, c) Symmetrical ideal transformer equivalent circuit, d) Ideal transformer equivalent circuit.

We now compute the scattered power in each symmetrical mode. For the first mode

$$\textit{Total Field} \;=\; \textit{Incident Field} + \textit{Scattered Field} \qquad \text{(C18)}$$

$$a_1' + b_1' \;=\; a_1'^0 + b_1'^0 + \textit{Scattered Field} \qquad \text{(C19)}$$

$$a_1' + a_1'\Gamma_{in1} = a_1' + a_1'\Gamma_{oc}\, e^{-j\frac{\pi}{2}\frac{1}{n}} + Scattered\ Field. \quad \text{(C20)}$$

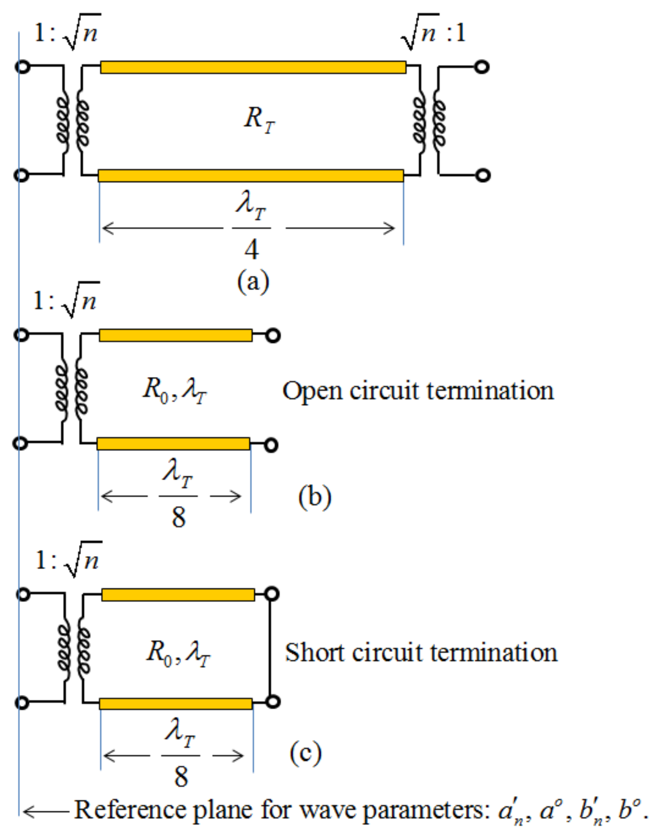

Fig. C2. Circuits for alternative modes eq. C17 incidence calculation: a) Symmetrical ideal former equivalent circuit for quarter wave dielectric slab, b) a'_1 mode incidence – open circuit bisection, c) a'_2 mode incidence – short circuit bisection.

Of course, the reflection coefficient corresponding to an open circuit, $\Gamma_{oc} = +1$. In order to compute Γ_{in1} the open circuit must be referred back through half the quarter wave transformer, a phase length of $\pi/4$. The open circuit reflection coefficient is therefore multiplied by $e^{-j\pi/2}$. This reflection coefficient corresponds to a normalized impedance $-j$ in the dielectric. The ideal transformer of the symmetrical equivalent circuit with ratio $1{:}\sqrt{n}$ corresponding to the transition from dielectric to free space characteristic impedance produces a normalized input impedance $-j/n$. It follows that the input reflection coefficient

$$\Gamma_{in1} = \frac{-\frac{j}{n} - 1}{\frac{-j}{n} + 1} = \frac{j|\tau_1| + 1}{j|\tau_1| - 1}. \quad \text{(C21)}$$

Substituting in the above prescription for the scattered field in mode 1, we obtain

$$\frac{a_1}{\sqrt{2}}\left[\frac{j|\tau_1| + 1}{j|\tau_1| - 1} - e^{-j\frac{\pi}{2}|\tau_1|}\right] = Scattered\ Field. \quad \text{(C22)}$$

The scattered power in mode 1 is therefore

$$P_{s1}^{'S} = \frac{|a_1|^2}{2}\left|\frac{j|\tau_1| + 1}{j|\tau_1| - 1} - e^{-j\frac{\pi}{2}|\tau_1|}\right|^2. \quad \text{(C23)}$$

A precisely parallel calculation for the antisymmetric mode 2, $\Gamma_{sc} = -1$, produces

$$\Gamma_{in2} = \frac{+\frac{j}{n} - 1}{+\frac{j}{n} + 1} = \frac{j|\tau_1| - 1}{j|\tau_1| + 1} \quad \text{(C24)}$$

and the scattered power in mode 2

$$P_{s2}^{'S} = \frac{|a_1|^2}{2}\left|\frac{j|\tau_1| - 1}{j|\tau_1| + 1} + e^{-j\frac{\pi}{2}|\tau_1|}\right|^2. \quad \text{(C25)}$$

In view of the orthogonality of the symmetric and antisymmetric modes, the total scattered power is the sum of the scattered powers in each mode,

$$P_s^{'S} = P_{s1}^{'S} + P_{s2}^{'S} \quad \text{(C26)}$$

Although the two calculations of scattered power result in quite different algebraic forms, they produce identical numerical results for the total scattered power, Fig. C3. In more detail, $P_{s1}^{'S}$ makes by far the larger contribution to the scattered power. This is to be expected as the electric field is strong near the symmetry plane for this mode and therefore strongly influenced by the dielectric. The electric field is correspondingly weak near the symmetry plane for the antisymmetric mode and therefore is only weakly influenced by the dielectric.

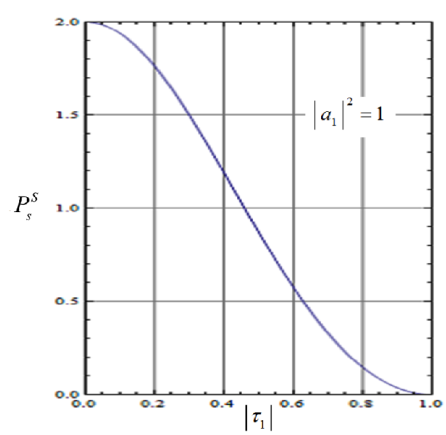

Fig. C3. Dielectric slab, normalized scattered power P_s^S.

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