On a Mathematical Theory of Open Metal–Dielectric Waveguides

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Abstract—Existence of symmetric waves in open metal–dielectric waveguides, a dielectric rod and the Goubau line, is proven by analyzing the of functional properties of the dispersion equations (DEs) and parameter-differentiation method, applied to the analytical and numerical solution of the DEs. Various limiting cases are investigated. Reduction to singular Sturm–Liouville boundary eigenvalue problems on the half-line is performed. Principal and higher-order surface waves are investigated.

Index Terms—Goubau line, Surface wave, Dispersion equation.

I. INTRODUCTION

Analysis of wave propagation in open metal–dielectric waveguides constitutes an important class of vector electromagnetic problems. A conducting cylinder covered by a concentric dielectric layer (see Fig. 1) is the simplest type of such guiding structures. The first studies date back to A. Sommerfeld who wrote that, "Taking into account the finite conductivity of a straight, round wire, a possible solution of Maxwell’s equations is a surface wave that travels along the wire and is non-radiating. This will be a TM wave at least consisting of the fundamental mode” [1]. Harms [2] was the first to obtain a dispersion equation (DE) for a conducting wire surrounded by a dielectric sheath. This structure was later called the Goubau line (GL) [3]. A thin dielectric cover in the Goubau line plays the role of conductivity and this line allows to transmit electromagnetic waves along a single wire when non-radiating surface waves are bound to the wire. Since the early 1950s, a wide variety of studies and applications of Goubau lines have been performed and published. We note, for instance, paper [4] dealing with theoretical and experimental investigations of surface waves along coated and corrugated conductors and guides with capacitive surface impedance, and book [5] which summarizes some aspects of the theory of surface waves. A review of different types of Goubau-type surface-wave transmission lines and a summary of the earlier results of Sommerfeld and Harms-Goubau lines are presented in [6], as well as the DEs for surface waves and various known applications. Mathematical and numerical investigations of the mode spectrum of Goubau lines are performed in [7-9]. A comprehensive list of references of more recent results can be found e.g., in [10, 11]. Several recent applications of Goubau lines are reported in [12, 13]. In spite of a long history of the studies partially reviewed above, many basic problems that have been solved for empty shielded waveguides (existence of normal waves, discreteness and localization of their spectrum on the complex plane, and so on) still remain unsolved for open metal–dielectric waveguides and even for the Goubau line. In particular, to the best of our knowledge a satisfactory mathematical analysis of multi-parameter DEs has yet to be carried out. This fact has become a driving force for us to carry out a systematic study of the corresponding mathematical issues using the recently obtained fundamental results in the mathematical theory of electromagnetic wave propagation [14, 15]. In these works, it is shown in particular that the spectrum of a broad family of metal-dielectric waveguides consists of isolated points on the complex plane (continuous spectrum is absent) and is localized in a strip; the analysis is based on the operator spectral theory methods.

In this paper we consider two basic structures: a dielectric waveguide of circular cross-section and the Goubau line. We reduce the problems of normal waves to the singular Sturm–Liouville boundary eigenvalue problems on the half-line with a discontinuous (piecewise constant) coefficient. It should be noted that the boundary conditions in the latter problem are complex functions of the spectral parameter and therefore there are no results concerning the spectrum existence and distribution in the classical literature on Sturm–Liouville boundary value problems. In such cases, when the boundary operator depends in a complicated manner on the complex spectral parameter, each Sturm–Liouville boundary eigenvalue problem or a family of such problems should be investigated separately using the analysis of this particular dependence. The determination of eigenvalues is reduced to certain nonlinear equations \( F(z, \vec{a}) = 0 \), called the DE for normal waves. Here, \( z \) is the spectral parameter (the longitudinal wavenumber of the running wave) and \( \vec{a} = (a_1, a_2, \ldots) \) is the vector of nonspectral parameters of a particular problem. DE specifies \( z = z(\vec{a}) \) as an implicit function. In many practical examples, only one nonspectral parameter is varied. We prove the existence of roots of DE by reducing it to the form \( F_1(z, a_1) = F_2(z) \), where function \( F_2(z) \) is characterized by infinite number of singular points and takes all real values on the intervals between these points, and the left-hand side is a continuous function of a fixed sign on a certain interval. Using the method of the parameter differentiation, DEs are reduced to the Cauchy problems to find an implicit function \( z = z(a_1) \). The determination of the propagation constants of symmetric waves propagating in the Goubau line with a thin, perfectly conducting cylinder.
placed inside a dielectric rod and with a thin dielectric layer are considered when the radius of the cylinder (or the width of the thin layer) is taken as a small parameter \((a_1)\). In its mathematical completeness including the development of the Sturm-Liouville theory, the present study is the first and necessary step towards the analysis of dielectric and metal–dielectric waveguides (fibers, Goubau lines) including complex waves, resonance and interaction phenomena, and structures with more complicated coating formed by linear and nonlinear layered or radially inhomogeneous media.

II. STATEMENT

Consider the propagation of symmetric eigenwaves, described in terms of nontrivial solutions to homogeneous Maxwell’s equations, in the dielectric rod with a radius \(a\) and the Goubau line, in which \(a, b, b > a\), are the radii of the internal (perfectly conducting) and external (dielectric) cylinders. Longitudinal wavenumbers \(k_s\) of the symmetric waves are determined as

\[
k_s^2 = \begin{cases} k_0^2 - \beta^2, & r > a, \\ k_0^2 - \beta^2, & r < a. \end{cases} \quad \text{(dielectric rod)}
\]

\[
k_s^2 = \begin{cases} k_0^2 - \beta^2, & r > b, \\ k_0^2 - \beta^2, & a < r < b. \end{cases} \quad \text{(Goubau line)}
\]

Here, \(\beta\) is the wave propagation constant (spectral parameter), \(\epsilon > 1\) is relative permittivity of the homogeneous dielectric, and \(k_0\) is the free-space wavenumber. We consider symmetric azimuthally-independent waves, having the nonzero components

\[
\mathbf{H} = [0, H_2(r, z), 0], \quad \mathbf{E} = [E_1(r, z), 0, E_3(r, z)],
\]

\[
E_1 = -\frac{i\beta}{k_s^2} \frac{d\phi}{dr} e^{-i\beta z}, \quad E_3 = \phi(r) e^{-i\beta z},
\]

\[
H_2 = -\frac{i\omega\mu}{k_s^2} \frac{d\phi}{dr} e^{-i\beta z},
\]

where \(\phi\) is a cylindrical function satisfying the Bessel equation:

\[
\mathcal{L}\phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) + k_s^2 \phi = 0.
\]

Determination of surface waves can be reduced to the following singular Sturm–Liouville boundary eigenvalue problems on the half-line with a discontinuous (piecewise constant) coefficient in the differential equation:

\[
\mathcal{L}\phi = 0, \quad r > 0, \quad |\phi|_{r=a} = \left[ \frac{\epsilon}{k_s^2} \frac{d\phi}{dr} \right]_{r=a} = 0, \quad (4)
\]

\[
\phi(r) \to 0, \quad r \to \infty,
\]

\[
\phi \in C^1[0, +\infty) \cap C^2(0, a) \cap C^2(a, +\infty)
\]

for the dielectric rod; and

\[
\mathcal{L}\phi = 0, \quad r > a, \quad \phi(a) = 0, \quad |\phi|_{r=b} = \left[ \frac{\epsilon}{k_s^2} \frac{d\phi}{dr} \right]_{r=b} = 0, \phi(r) \to 0, \quad r \to \infty,
\]

\[
\phi \in C^1[a, +\infty) \cap C^2(a, b) \cap C^2(b, +\infty)
\]

for the Goubau line.

The conditions at infinity in (4) and (6) and the form (2) can be taken as a definition of symmetric (surface) waves in open waveguides under study. The surface waves are described in terms of real-valued quantities; in particular, the boundary operators in (4) and (6) are defined (as real-valued functions of a real variable \(\gamma\) or \(\lambda = \gamma^2\)) on a certain interval \(I\). The spectrum of surface waves may be empty, or they may consist of several (real) points located on this interval. The results of the classical Sturm–Liouville theory concerning existence and distribution of the (real) spectrum are not applicable here, and we will perform in this work the corresponding analysis from the very beginning by reducing the boundary eigenvalue problems under study to DEs. This will, in turn, enable us to prove all the required statements and determine the eigenvalues numerically or analytically.

Make the following designations:

\[
\gamma = \frac{\beta}{k_0}, \quad x = k_0 a \sqrt{\epsilon - \gamma^2}, \quad u = k_0 a \sqrt{\epsilon - 1}, \quad s = \frac{b}{a} > 1.
\]

For surface waves that must decay at infinity, potential function \(\phi(r)\) is represented in terms of cylindrical functions that tend to zero at infinity, so that

\[
\phi(r) = \begin{cases} AK_0\left(\frac{\sqrt{\epsilon-1}}{a} x\right), & r > a, \\ BJ_0\left(\frac{\sqrt{\epsilon-1}}{a} x\right), & r < a, \end{cases}
\]

for the dielectric rod, and

\[
\phi(r) = \begin{cases} AK_0\left(\frac{\sqrt{\epsilon-1}}{a} x\right), & r > b, \\ BJ_0\left(\frac{\sqrt{\epsilon-1}}{a} x\right) [J_0(\sqrt{s} a x) - Y_0(\sqrt{s} a x) J_0(x)] , & a < r < b. \end{cases}
\]

for the Goubau line.

Here, \(J_0(x), Y_0(x), K_0(x),\) and \(K_1(x)\) denote, respectively, the Bessel and Neumann functions of the order \(k = 0, 1\) and the McDonald functions.
III. DIELECTRIC ROD

By applying to (9) the condition of continuity of $\phi$ and its derivative at the point where the coefficient is discontinuous, we obtain the desired DE in the form [18]

$$ \frac{J_0(x)}{J_1(x)} = \epsilon \sqrt{u^2 - x^2} \frac{H_0^{(1)}(i \sqrt{u^2 - x^2})}{H_1^{(1)}(i \sqrt{u^2 - x^2})}, \quad (11) $$

which will be considered in what follows with respect to new variable $x$ (instead of $\beta$). Here, $H_0^{(1)}(\cdot)$ denotes the zero-order Hankel function of the first kind.

We rewrite (11) as

$$ G_d(x, u) = F_d(x), \quad (12) $$

$$ F_D(x, u) = 0, \quad F_D(x, u) \equiv G_d(x, u) - F_d(x), \quad (13) $$

where

$$ G_d(x, u) = -\epsilon \sqrt{u^2 - x^2} \frac{K_0(\sqrt{u^2 - x^2})}{K_1(\sqrt{u^2 - x^2})}, \quad (14) $$

$$ F_d(x) = x \cot j(x), \quad \cot j(x) := \frac{J_0(x)}{J_1(x)} \quad (15) $$

In view of numerical solution and analysis of the limiting cases $a \to 0$ and $\epsilon \to 1$, it is reasonable to consider equation (12) with respect to the new independent variable

$$ w = \sqrt{u^2 - x^2} = \sqrt{\gamma^2 - 1} > 0 $$

with

$$ x = k_0 a \sqrt{\epsilon - 1 - w^2}, \quad x \in (0, \sqrt{\epsilon - 1}) $$

where $\epsilon \geq 1$.

Equation (13) determines $x = x(u)$ as an implicit function of the parameter $u > 0$, and can be used to prove that this function exists for all $u > u^*$, where $u^* > \mu_1^{(0)}$. Namely, using the properties of the cylindrical cotangent and function $G_d(x, u)$, we arrive at the known result [2], which is:

If

$$ u = k_0 a \sqrt{\epsilon - 1} > \mu_1^{(0)}, $$

then there exists a root

$$ x = x^* \in (\mu_1^{(0)}, \mu_1^{(1)}) $$

of DE (13). If

$$ u \in (\mu_1^{(0)}, \mu_1^{(1)}), $$

then (13) has only one root.

IV. GOUBAU LINE

A. Dispersion equation

By applying the transmission (continuity) condition to (10) we obtain the desired DE for surface waves in the Goubau line in terms of variables (8), which reads

$$ F_G(x, u, s) \equiv G_g(x, u, s) - F_g(x, s) = 0, \quad (16) $$

where

$$ G_g(x, u, s) = \epsilon \sqrt{u^2 - x^2} \frac{K_0(s \sqrt{u^2 - x^2})}{K_1(s \sqrt{u^2 - x^2})}, \quad (17) $$

$$ F_g(x, s) = \frac{\Phi_1(x, s)}{\Phi_2(x, s)}, \quad (18) $$

$$ \Phi_1(x, s) = J_0(s x) Y_0(x) - J_0(x) Y_0(s x), $$

$$ \Phi_2(x, s) = J_0(x) Y_1(s x) - J_1(s x) Y_0(x). $$

B. Principal surface wave

Let us first prove the existence of the principal (fundamental) surface wave for all $u > 0$ and $s > 0$. We denote by $y_m^{(k)}$ the zeros of the Neumann functions $Y_k(x)$ ($k = 0, 1$). According to [17], the zeros of the Bessel and Neumann functions alternate:

$$ \mu_0^{(1)} = 0, \quad 0 < y_1^{(0)} < y_1^{(1)} < \mu_1^{(0)} < \mu_1^{(1)} < \cdots $$

Denote by $h_m^0(q)$, and $h_m^{10}(q)$ ($m = 1, 2, \ldots$) zeros of the products of cylindrical functions

$$ Z_{00}(q, y) = J_0(q y) Y_0(y) - Y_0(q y) J_0(y), $$

$$ Z_{10}(q, y) = J_1(q y) Y_0(y) - Y_1(q y) J_0(y). $$

For a fixed $q > 1$, all these zeros are real and simple points [17]; the sequences $\{h_m^0(q)\}$ and $\{h_m^{10}(q)\}$ are increasing w.r.t the index $m$, and the zeros of their terms alternate on the positive semi-axis. The $h_m^{10}(q)$ are decreasing functions of $q$, for $q > 1$.

Thus, $F_g(x, s)$ in (18), can be represented as

$$ F_g(x, s) = x \cot j(x), \quad \cot j(x) = \frac{Z_{00}(s, x)}{Z_{10}(s, x)}, \quad (21) $$

Fig. 2: Graphs of $G_d(x, u)$ (——) and $F_d(x)$ (- - -) against $x$ for $\epsilon = 5$, $k_0 a = 2.8$, and $u = 5.6$ illustrating (graphical) solution of DE (12) for the dielectric rod.

One may call function $\cot j(x)$ the cylindrical cotangent, because it exhibits all basic properties of the trigonometric cotangent; namely, $\cot j(x)$ takes all real values on every interval $(\mu_0^{(k)}, \mu_1^{(k+1)})$, where $\mu_m^{(k)}$ are zeros of cylindrical functions $J_k(x)$ ($k = 0, 1$), and $\cot j(x) > 0$ $x \in (\mu_0^{(k)}, \mu_1^{(k+1)})$, $k = 1, 2, \ldots$. 
The domain of function $F_G(x, u, s)$ is

\[ D_{F_G}(u, s) = \{ x : \text{ } | x | \leq u \triangleleft h_G(u, s) \}, \quad u > 0, \quad s \geq 1, \]

\[ H_G(u, s) = \{ h_{01}^m(s) : 0 \leq h_{01}^m(s) \leq u, \quad m = 1, 2, \ldots N(u) \}, \]

\[ H_G(u, s) = \emptyset \quad (N(u) = 0) \]

if all $h_{01}^m(s) > u$ \quad ($m = 1, 2, \ldots N(u)$).

According to the conditions on parameters we will consider $F_G(x, u, s)$ on the set

\[ D_{F_G}(u, s) = \{ (x, u, s) : 0 \leq x \leq u \triangleleft h_G(u, s), \quad u > 0, \quad s > 1 \}. \]

$F_G(x, u, s)$ is a continuously differentiable function of $x$ on $D_{F_G}(u, s)$ for all $u > 0$ and $s > 1$. Below we will prove that $N(u) \geq 0$ specifies the number of surface waves propagating in the Goubau line for the given $u > 0$.

From the explicit form of functions in (16) it follows that for all $u > 0$ and $s > 1$: (i) $F_4(x, s)$ takes, as a function of $x$, all positive values on the interval $I(s) = (0, h_{01}^0(s))$; and (ii) $G_4(x, u, s)$, as a function of $x$, all values from the interval $I_T(s, u) = (0, T(s, u))$ where

\[ T(s, u) = c u K_0(s u) K_1(s u) > 0. \]

Consequently, there exists (at least) one root $x^0 = x^0(s, u) \in I(s)$ of equation (16). This root corresponds to the fundamental surface wave, which exists for all $u > 0$. The analysis is clearly illustrated in Figs. 5 and 6, where the $x$-coordinate of the point of intersection on the interval between the origin and the first vertical asymptote is the root $x^0$ corresponding to the fundamental (no-cut-off) surface wave.

Check the location of roots for large values of $s$. Using explicit form (20), one can show that, for $s > 1$, there exist two zeros of $\Phi_2(x, s)$ when

\[ \frac{y_k(1)}{s} < y_1(0), \quad \frac{y_k(0)}{s} < y_1(0), \quad k = 1, 2, \ldots N. \]  

In particular,

\[ N = 1: \quad s > \frac{y_2(0)}{y_1(0)} = 4.333; \quad N = 2: \quad s > \frac{y_3(0)}{y_1(0)} = 7.89. \]

However, using the properties of the function that enter (17) and (18), one cannot say anything about the existence of roots in the interval $(0, \frac{y_0}{x})$. I will consider this interval below by reducing the examination to the analysis of the Cauchy problem.

Sufficient conditions for the existence of two zeros of $\Phi_1(x, s)$ can be obtained in a similar manner

\[ \frac{y_2(0)}{s} < y_1(0). \]  

Thus, using (19) and (23), we find that when

\[ s > \max \left\{ \frac{y_2(0)}{y_1(0)} \left( \frac{y_2(1)}{y_1(1)} \right) \right\} = \max \{ 7.930044, 6.0766 \} = 7.930044, \]
the zeros of the numerator and denominator in (18) are disposed as follows:

$$0 < x_1^{\text{den}} < x_1^{\text{num}}, \quad x_2^{\text{den}} < x_2^{\text{num}}.$$  

Applying the properties of function $\cot_j (x)$ similar to those of $\cot x$ (zeros and singular points of $\cot_j (x)$ alternate, each zero is followed by a singular point, and the function takes all real values between the neighboring singular points), one can show that for

$$u > \frac{y_2(0)}{s} \quad \text{or} \quad k_0 b \sqrt{e - 1} > y_2(0) = 3.957678$$

and under the condition (24); that is, when

$$k_0 a < \frac{k_0 b}{7.930044},$$

there exists at least one root $x = x_g$ of equation (16) located in the interval $(x_1^{\text{den}}, x_1^{\text{num}})$.

C. Higher-order waves, limiting cases and the parameter differentiation method

The analysis of Section 2.3 is not applicable when $s \to 1$. Therefore, we present an alternative approach based on parameter differentiation and implicit-function methods that enables one to consider various limiting cases.

In (16), we introduce the change of variables

$$sx = y, \quad v = su, \quad t = \frac{1}{s}.$$

If $b$ is fixed, then $t = \frac{a}{b} \to 0$ if $a \to 0$. Rewrite equation (16) in new variables:

$$\tilde{F}_G(y, t) = \tilde{G}_o(y, v) - \tilde{F}_y(y, t) = 0,$$

where

$$\tilde{G}_o(y, v) = \epsilon \sqrt{v^2 - y^2} \frac{K_0(\sqrt{v^2 - y^2})}{K_1(\sqrt{v^2 - y^2})},$$

$$\tilde{F}_y(y, t) = \frac{\tilde{\Phi}_1(y, t)}{\tilde{\Phi}_2(y, t)}.$$

One can show that for $t \to 0$, when the Goubaue line turns into a dielectric rod, equation (16) in the form (25) turns into DE (12) for symmetric waves of the dielectric rod.

The roots $y = y(t) = k_0 b \sqrt{e - \gamma^2}$ of (25) can be found as solutions to the Cauchy problem

$$\{ \begin{array}{ll}
\frac{dy}{dt} = \tilde{\Phi}_G(y, t), & y(0) = y_0,
\hat{\Phi}_G(y, t) = - \frac{\partial \tilde{F}_G(y, t)}{\partial y} \frac{dy}{dt},
\end{array} \tag{26}$$

where $y_0 \in (0, k_0 b \sqrt{e})$ is any solution of equation (12). Taking into account the recurrence relations for cylindrical functions and the asymptotic behavior $Y_0(z) \sim \frac{2}{\pi} \ln z$ as $z \to 0$ one can show that

$$Q(y_0) = \frac{\partial \tilde{G}_o(y_0, 0)}{\partial y} + \frac{J_0(y_0)}{J_1(y_0)} - y_0^2 \frac{J_1^2(y_0)}{J_0(y_0)} - J_1(y_0).$$

The Cauchy problem (26) is solvable under the condition

$$Q(y_0) \neq 0,$$

which provides also that $\tilde{\Phi}_G(y, t)$ is continuous in the vicinity of the point $(y, t) = (y_0, 0)$.

Thus placing an infinitely thin ($a \sim 0$), perfectly conducting rod into a dielectric fiber ($a = 0$) creates a regular perturbation for the spectrum of eigenwaves of the dielectric fiber (rod).

Let us examine another limiting case $b - a \to +0$, or $s \to +1$, when the Goubaue line turns into a perfectly conducting cylinder with radius $b$. In the limit $s \to +1$, (16) takes the form

$$\sqrt{u^2 - x^2} K_0(\sqrt{u^2 - x^2}) K_1(\sqrt{u^2 - x^2}) = 0,$$

or

$$H_0^{(1)}(\sqrt{x^2 - u^2}) = 0.$$

Hence, the propagation constants

$$\gamma = \gamma_m^{(1)} = \sqrt{1 - \left(\frac{z_m^{(1)}}{k_0 a}\right)^2,} \tag{27}$$

are complex. Here, $H_0^{(1)}(z_m^{(1)}) = 0$ where $z_m^{(1)}$ denotes a (complex) zero of the Hankel function $H_0^{(1)}(z)$ with $\Re z_m^{(1)} < 0$ ($m = 1, 2, \ldots$).

Other solutions in this limiting case can be obtained from the equation

$$x^2 - u^2 = 0, \quad \gamma = 1.$$

Implicit function $x = x(s)$ can be determined as a solution to the Cauchy problem

$$\{ \begin{array}{ll}
\frac{dx}{ds} = \tilde{\Psi}_G(x, s), & \tilde{\Psi}_G(x, s) = - \frac{\partial \tilde{F}_G(x, s)}{\partial x} \frac{dx}{ds},
\end{array} \tag{28}$$

which is solvable under the condition

$$\frac{\partial \tilde{F}_G(x, s)}{\partial x} |_{x = u, s = 1} \neq 0$$

and $\tilde{\Psi}_G(x, s)$ is continuous in the vicinity of the point $(x, s) = (u, 1)$. We have

$$\frac{\partial \tilde{F}_G(x, s)}{\partial s} |_{x = u, s = 1} = -u^2,$$

$$\frac{\partial \tilde{F}_G(x, s)}{\partial x} = Q_1,$$

$$Q_1 \sim x e \left[2 s \ln(s \sqrt{u^2 - x^2}) + O(1)\right] \to -\infty, \quad x \to -u.$$

So, the Cauchy problem (28) is not solvable in the vicinity of the point $(x, s) = (u, 1)$.

Thus, again we see that, generally, introduction of a thin dielectric layer is a not regular perturbation for the spectrum of eigenwaves produced by TEM wave ($\gamma = 1$) or a decaying wave that corresponds to complex propagation constant $\gamma_m^{(1)}$. Below we will overcome this difficulty by replacing a singular form of the DE given in (25), by a regular one.
In order to prove the existence of a solution \( x = x^0 = x^0(s) \) to DE (16) in the vicinity of the point \( x = u \) corresponding to the principal surface wave consider the limiting case \( s \to +1 \) and rewrite equation (16) in the regular form

\[
F_0(x, u, s) \equiv G_0(x, u, s)\Phi_2(x, s) - x\Phi_1(x, s) = 0, \quad 0 < x < u. \tag{29}
\]

We have \( F_0(x = u, u, 1) \equiv 0 \) for all \( u > 0 \) and \( \frac{\partial F_0(x, u, 1)}{\partial x} \neq 0 \) (e.g. \( F_0(x, u, 1) \approx 3.6538 \) at \( u = 3.999 \ldots \) and \( s = 1 \)). Consequently, \( x = x(s) = x^0(s) \) is a regular perturbation of the value \( x = u \) (or \( \gamma = 1 \)) attained at \( s = 1 \) and can be determined as a solution to the Cauchy problem

\[
\begin{cases}
\frac{dx^m}{ds} = \tilde{F}_0(x, u, s), \\
x^m(s^0_m) = u.
\end{cases}
\]

Note that here, \( u \) is a fixed positive number.

Let us now consider the higher-order surface waves. Note first of all that: (i) for every \( u > 0 \) there exists a (minimal positive) zero \( h^0_m(s) \) of function \( \Phi_1(x, s) \) such that

\[
u = h^0_0(s^0_m) \quad \text{for one and only one} \quad s = s^0_m = s(u) > 1, \quad m = 1, 2, \ldots ; \tag{30}
\]

(ii) according to the explicit form (29) of the DE \( F_0(x = u, u, s^0_m) = 0 \) for any \( u > 0 \) and \( \frac{\partial F_0(x, u, s)}{\partial x} \neq 0 \). Numbers \( s^0_m \) can be determined, for every given \( u > 0 \), by numerically solving the equation

\[
\Phi_1(u, s) = 0. \tag{31}
\]

Thus \( x = x^m(s) = x^m(s) \) can be determined as a regular perturbation of the value \( x = h^0_0 \) attained at \( s = s^0_0 \) for a fixed \( u > 0 \), by finding the implicit function \( x^m = x^m(s) \) as a (real) solution to the Cauchy problem similar to (28)

\[
\begin{cases}
\frac{dx^m}{ds} = \tilde{F}_0(x, u, s), \\
x^m(s^0_m) = h^0_0(s^0_m) \quad \text{for one and only one} \quad s = s^0_m, \quad m = 1, 2, \ldots N_0(u). \tag{32}
\end{cases}
\]

Note that \( x^m(s) \) satisfies the condition (see Fig. 8)

\[
h^0_0(s) > x^m(s) > h^0_1(s), \quad m = 1, 2, \ldots N_0(u). \tag{33}
\]

We can verify the solvability conditions:

\[
\frac{\partial F_0(x, u, s)}{\partial x} \bigg|_{x=h^0_m(s), s=s^0_m} \neq 0
\]

and \( \tilde{F}_0(x, u, s) \) is continuous in the vicinity of point \( A^m_0 \) \( (x^m = h^0_0(s), \quad s = s^0_0) \). Calculating the derivative with respect to \( x \) at point \( A^m_0 \) where \( F_0(x(s^0_0), u, s^0_0) = 0 \), we have

\[
\frac{\partial F_0(x, u, s)}{\partial x} \bigg|_{x=h^0_m(s), s=s^0_m} = \frac{\partial G_0(x, u, s)}{\partial x} \bigg|_{x=h^0_m(s), s=s^0_m} = \Phi_2(h^0_m(s), s^0_m) - h^0_0(s^0_m) \left( \Phi_1(h^0_m(s), s^0_m), s^0_m \right) = Q_{0m}(u).
\]

Calculating the derivative with respect to \( s \) at the same point, and taking into account the identity

\[
\frac{\partial \Phi_1}{\partial s} = x\Phi_2(x, s),
\]

we obtain

\[
\frac{\partial F_0(x, u, s)}{\partial s} \bigg|_{x=h^0_m(s), s=s^0_m} = G_0(h^0_0(s^0_m), u, s^0_m) \Phi_2(h^0_0(s^0_m), s^0_m) = P_{0m}(u).
\]

All functions that enter the expression for \( \frac{\partial F_0(x, u, s)}{\partial s} \) are continuous in the vicinity of \( A_0 \) and \( P_{0m}(u) \) is a finite quantity; therefore, \( \frac{\partial F_0(x, u, s)}{\partial s} \) is continuous at \( A^m_0 \). Thus, if a given \( u = k_0\alpha\sqrt{\varepsilon - 1} \) satisfies the conditions

\[
Q_{0m}(u) \neq 0, \quad u = k_0\alpha\sqrt{\varepsilon - 1} > h^0_0(s),
\]

where, for a given \( s > 1 \), \( h^0_0(s) \) is the \( m \)th root of the function

\[
\Phi_2(x, s) \equiv J_0(sx)Y_1(x) - J_1(x)Y_0(sx),
\]

the Cauchy problem (32) is uniquely solvable.

Thus, we have demonstrated that, for arbitrary \( s = b/a > 1 \) that may be sufficiently close to unity, there exists a root \( x^1 = x^1(s) \) of DE (16) (more specifically, implicit function \( x^1(s) \) specified by equation (16) in the vicinity of point \( A^1_0 \)). In other words, for an arbitrarily thin dielectric layer there exists (at least one) surface wave propagating in the Goubau line under the condition

\[
u = k_0\alpha\sqrt{\varepsilon - 1} > h^0_0(s). \tag{34}
\]

Relationship (34) is the necessary condition that provides the existence of at least one such wave. Taking into account the domain of function \( F_0(x, u, s) \) and \( \tilde{F}_0(x, u, s) \) we have the condition \( x \in (0, k_0\alpha\sqrt{\varepsilon}) \).

V. CROSS-SECTIONAL STRUCTURE OF HIGHER-ORDER WAVES

The cross-sectional structure of higher-order waves, or numbers of zeros (oscillations) of potential function \( \phi(r) \), can be determined by using the classical Sturm–Liouville oscillation theory which may be a subject of a future separate study. In this paper, we prove the necessary mathematical items concerning the structure of higher-order waves independently using the analysis based on explicit expressions of the functions entering the DE. The result is: if \( \{x_i\}^K_{i=1} \subset [0, u] \) are \( K \) real zeros of function (16) w.r.t. \( x \), then \( K = n \) where \( n = 1, 2 \ldots \) is the sequence index of potential function \( \phi_n(r) \). For \( \lambda \) going to infinity, the number of zeros of a \( \phi_n(r) \) defined on \((a, b)\) also goes to infinity, while the intervals between two neighboring zeros contained in the interval tend to zero.

Figure 7 illustrates this behavior for the first four potential functions. We see that the potential function corresponding to the principal wave (superscript zero) has no oscillations on the interval covering the cross section of the dielectric layer, while the potential functions corresponding to higher-order waves (superscripts 1, 2, and 3) have, respectively, 1, 2, and 3 oscillations on this interval.
Fig. 5: Graphs show the location of the first roots of DE (16) w.r.t. $x$ for $u = 4$ and $s = s_0^1 = 1.78$: the first three roots are close to the zeros of function $\Phi_2$ corresponding to vertical asymptotes with $u = h_0^1(s_0^1)$ where $\Phi_1(s_0^1, u) = 0$. The root $x_0(s_0^1)$ of DE (the first zero of $F_0$) corresponds to the principal surface wave and is close to the first zero of $\Phi_2(x, s_0^1)$ (the 1st vertical asymptote of function $F_0$), the second root $x_1(s_0^1) = u$ corresponds to the first higher-order wave and coincides with the first zero $h_0^1(s_0^1)$ of function $\Phi_1$.

Fig. 6: Location of the first four roots of DE (16) w.r.t. $x$ for $u = 4$ and $s = 3.4$: the roots are close to the zeros of function $\Phi_2$, corresponding to vertical asymptotes on the graph.

VI. RESULTS OF CALCULATION OF SURFACE WAVES IN THE GOUBAU LINE

By solving numerically the Cauchy problems (28) and (32) we calculate the normalized propagation constants $\gamma$ of the fundamental (with no oscillations) and higher-order surface modes of the Goubau line vs. parameter $s$ for different values of parameter $u$. The number of zeros of potential functions $\phi(r)$ corresponds to the index of a higher-order wave: the cutoff values of $s$ are determined by condition (2). The results are summarized in Fig. 8; it can be seen that the potential (eigen) functions preserve their types (number of zeros or oscillations) along the curves on the phase plane $(s, \gamma)$.

Tables 1 and 2 exemplify some numerical values of the normalized propagation constants of the GL surface modes calculated for normalized (dimensionless) values of the geometrical parameters. The number of decimals show the accuracy of calculation using numerical solution of the Cauchy problems.

Tab. 1: Roots $x$ of DE (29) and the corresponding values of $\gamma$ at $a = k_0a = 0.2$ and $\epsilon = 2$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\epsilon$</th>
<th>$s$</th>
<th>$x$</th>
<th>$\gamma$</th>
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Tab. 2: Roots $x$ of DE (29) and the corresponding values of $\gamma$ at $a = 1$ and $\epsilon = 2$

<table>
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<th>$s$</th>
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VII. CONCLUSION

In this work, we have shown that a principal surface mode in the Goubau line exists for arbitrary values of permittivity of the cladding and radius of the inferred conductor (wire). If the electrical cross-sectional dimension of the line is sufficiently large, \( k_0 a \sqrt{\varepsilon - 1} > h_0^1(1) \) where \( h_0^1(1) \) is an absolute constant, then there exist several higher-order GL surface modes, each having a distinct cross-sectional structure in the dielectric layer; namely, the number of zeros (oscillations) of the potential function corresponding to the principal and the \( n \)th higher-order GL mode equals the index of the wave: \( n \) for the principal and \( n \) for a higher-order mode.

The longitudinal wavenumbers of the higher-order modes can be efficiently calculated by deriving the numerical solution of a Cauchy problem obtained by the parameter differentiation method.

We have also reduced the determination of the spectrum of symmetric surface modes to singular Sturm-Liouville boundary eigenvalue problems on the half-line. The latter is reduced to the solution of (16) DEs involving the cylindrical tangent and cotangent functions, and then to the determination of the parameter dependence of the DE zeros using analytical and numerical solution to the Cauchy problems.

Several limiting cases have been investigated. We show that when an infinitely thin, perfectly conducting rod is placed into the cross-sectional origin of a dielectric fiber, it creates a regular perturbation of the spectrum of surface waves of the Goubau line with respect to the surface wave spectrum of the fiber. The spectrum of surface waves of the Goubau line with an infinitely thin dielectric coating is a perturbed set of zeros of a well-defined family of functions.

The present approach remains valid for a GL with a lossy dielectric layer (complex permittivity parameter). In that case it is necessary to investigate all functions entering the DEs as functions of one or two complex variables and apply the methods of searching for complex roots of the DEs. This is a subject of our future study.

REFERENCES


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